Chapter 1

Introduction

The uniform empirical process central limit theorem (CLT) and law of the iterated logarithm (LIL) for i.i.d. observations has been the subject of extensive study, but much less is known for dependent observations. Recent work which improves on this situation includes [2], [3], [4], [7] and [8], where Levental studies the uniform CLT for Markov chains, and where Doukhan, Massart, and Rio [4], and Arcones [2] and Yu [3] obtained the uniform CLT and LIL for stationary processes satisfying various mixing conditions. Our main effort has been to generalize Levental's uniform CLT results for Markov chains from the family of uniformly bounded functions to various families of unbounded functions, and to prove the uniform LIL for Markov chains. Although application of the uniform CLT and LIL for strictly stationary processes satisfying various mixing conditions yields results of this type, our point of view was that a direct approach should yield better results for the Markov case. Examples are given to show the differences, and completely substantiate this point of view.

Let $(S, \mathcal{G}, P)$ be a probability space and let $\mathcal{F}$ be a set of measurable functions on S with an envelope function F finite everywhere. Let $X_1, X_2, ...$ be a strictly stationary sequence of random variables with distribution P, and define the empirical measures P_n, based on $\{X_i\}$, as $P_n = n^{-1}\sum_{i=1}^n \delta_{X_i}$. We say the uniform CLT holds over $\mathcal{F}$, if $n^{\frac{1}{2}}(P_n - P)$ converges in law, in the space $l^\infty(\mathcal{F})$ to a Gaussian process. $l^\infty(\mathcal{F})$ is the set of bounded functions on $\mathcal{F}$ with sup-norm. We say the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if there exists a compact set K in $l^\infty(\mathcal{F})$ such that, with probability one,

$$\left\{(2n\log\log n)^{-\frac{1}{2}}\sum_{j=1}^{n}\left(f(X_j)-Ef(X_1)\right):f\in\mathcal{F}\right\}_{n\geq 1}$$

is relatively compact and its limit set is K, and the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if, with probability one,

$$\sup_{n}\sup_{f\in\mathcal{F}}(2n\log\log n)^{-\frac{1}{2}}\left|\sum_{j=1}^{n}\left(f(X_j)-Ef(X_1)\right)\right|<\infty.$$

In Chapter 2 and Chapter 3 we study Markov chains with countable state spaces. Let $\{X_i\}_{i\geq 0}$ be a positive recurrent irreducible Markov chain taking values in $S=\{1,2,3,\cdots\}$ with the unique invariant probability measure π, N_i be the i-th hitting time of state 1,

$$m_{i,j}=E\left(\min\{n:n\geq 1,X_n=j\}\mid X_0=i\right).$$

Levental (1990) [8] proved that for Markov chains if $E(N_2-N_1)^2<\infty$, then the uniform CLT holds over $\{1_A:A\subseteq S\}$ if and only if

$$\sum_{k=1}^{\infty}\pi(k)m_{1,k}^{\frac{1}{2}}<\infty.$$

In Chapter 2 we prove a uniform CLT which generalizes Levental's theorem from the set of indicator functions to the set of possibly unbounded functions. This uniform CLT over the family of functions dominated by a non-negative function is the best possible result for positive recurrent irreducible Markov chains with a countable state space. Let F be a non-negative function on S and $\mathcal{F}=\{f:|f|\leq F\}$. We have that the uniform CLT holds over $\mathcal{F}$ if and only if $E(N_2-N_1)^2<\infty$,

$$E\Big(\sum_{N_1<j\leq N_2}F(X_j)\Big)^2<\infty \tag{1.1}$$

and

$$\sum_{k=1}^{\infty}F(k)\pi(k)m_{1,k}^{\frac{1}{2}}<\infty. \tag{1.2}$$

In Chapter 3 we prove the compact LIL and bounded LIL for Markov chains under a weaker condition than (1.2). Assume $E(N_2-N_1)^2<\infty$, (1.1) holds and choose a suitable order of S as indicated in Theorem 3.1 in Chapter 3. If

$$(\log\log n)^{-\frac{1}{2}}\sum_{k=1}^{n}F(k)\pi(k)m_{1,k}^{\frac{1}{2}}\to 0, \tag{1.3}$$

then the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$. Conversely, if the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ and there are $c,\alpha>0$ such that $\pi(k)\geq ck^{-\alpha}$ for all $k\in S$, then (1.3) holds. We also have the bounded LIL when (1.3) is replaced by

$$\sup_n(\log\log n)^{-\frac{1}{2}}\sum_{k=1}^{n}F(k)\pi(k)m_{1,k}^{\frac{1}{2}}<\infty.$$

In Chapter 4 and Chapter 5 we will deal with Markov chains in general state spaces. However, a uniform CLT and LIL is not always possible for families of functions of the form $\{f : |f|\leq F\}$ in the general state space setting. The following example illustrates this. In particular, it points out the need for additional assumptions on the family of functions.

Let $\{X_i\}$ be a i.i.d. sequence of random variables with distribution function $H(x)$ and measure μ. Let $\{x_1,x_2,...\}$ be the set of all jumps of $H(x)$ and λ be the measure with $\lambda(x_i)=\mu(x_i)$ for all i and $\lambda(\mathbf{R})=\sum_i\lambda(x_i)$. Assume $\lambda(\mathbf{R})<1$. Let $\nu=\mu-\lambda$ and $\delta=\lambda(\mathbf{R})>0$. Since $G(x)=H(x)-\lambda((-\infty,x])$ is increasing and continuous, we can choose $y_1,y_2,...$ such that $A_1=(-\infty,y_1]$, $\nu(A_1)=c$, ..., $A_k=(y_{k-1},y_k]$, $\nu(A_k)=ck^{-\frac{3}{2}}$ where $c=\delta/\sum_{k=0}^{\infty}k^{-\frac{3}{2}}$. Let

$$\mathcal{F}=\{1_A : A=\cup_{i\in I}A_i, I\subset\mathbf{N}\}.$$

Then by Theorem 2.7 in Chapter 2 and Theorem 3.1 in Chapter 3, we need at least

$$\sup_n\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}\sqrt{\mu(A_k)}<\infty$$

for the bounded LIL or the uniform CLT on $\mathcal{F}$. But

$$\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}\sqrt{\mu(A_k)}\geq\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}\sqrt{\nu(A_k)}=\frac{1}{\sqrt{\log\log n}}\sum_{k=1}^{n}\sqrt{c}k^{-\frac{3}{4}}$$

diverges as $n\to\infty$. Thus some restriction on the family of functions is required for the uniform CLT or LIL on a general state space.

We follow Levental's [7] approach to study uniform limit theorems over the functions in $\mathcal{F}$ provided $\mathcal{F}$ satisfies a combinatorial entropy condition due to Kolčinskiĭ [6] and Pollard [9]

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty, \tag{1.4}$$

where

$$N_2(\varepsilon, \mathcal{F}) = \sup_Q (N_2(\varepsilon, \mathcal{F}, Q)),$$

the sup is taken over all the measures on S with finite support, and $N_2(\varepsilon, \mathcal{F}, Q)$ is the minimum m for which there exists $g_1, ..., g_m$ in $L^2(Q)$ such that, for all $f \in \mathcal{F}$, $\| f - g_i \|_{L^2(Q)} < \varepsilon$, for some $1 \leq i \leq m$.

Dudley [5] proved the above combinatorial condition (1.4) is satisfied in the case where the subgraphs of the functions in $\mathcal{F}$ are a VC class of sets (see Section 4 of Chapter 4 for definition).

In Chapter 4 we improve Levental's results by extending the family of functions from uniformly bounded to the condition that its envelope function F is in L^2 and the CLT holds for F, and reduce the $2 + \delta$ moment condition of renewal time to ergodicity of degree 2.

Alexander and Talagrand [1] proved compact and bounded LIL's on VC classes of functions in the i.i.d. case with envelope function F satisfying

$$E\left(\frac{F^2(X)}{LLF(X)}\right) < \infty.$$

We will extend these results to Markov chains. Let $\{X_n\}$ be a Markov chain with ergodicity of degree 2 and order 1, and let $\mathcal{F}$ be a countable family of functions on S satisfying

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty,$$

and assume its envelope function F satisfies

$$\int_C \pi(dx) E_x \max_{n \leq \tau_C} \left(S_n^2(F)/LLS_n(F)\right) < \infty.$$

Here C is a small set and τ_C is the hitting time of C as defined in Chapter 4. If $n^{-\frac{1}{2}} S(f)$ converges weakly to $N(0, \sigma_f^2)$ for all $f \in \mathcal{F}$ and

$$\sup_{f \in \mathcal{F}} \sigma_f^2 < \infty$$

then the bounded LIL holds over $\mathcal{F}$.

The empirical process CLT and LIL for stationary sequences satisfying a mixing condition require the envelope function of the function class be in $L^p(P)$ for some $p > 2$. We give an example such that our conditions hold, but the envelope function $F \notin L^p(P)$ for all $p > 2$. Additional examples are also included to highlight other differences between our direct approach to the problem for Markov chains and that obtainable from [2], [3], and [4].

Bibliography

[1] Alexander, K. and Talagrand, M. The law of iterated logarithm for empirical processes on Vapnik-Cervonenkis Classes. J. of Multivariate Analysis 30,1989, 155-166.

[2] Arcones, M. A. The law of the iterated logarithm for empirical processes under absolute regularity. Journal of Theoretical Probability 8, no. 2, 1995, 433-451.

[3] Arcones, M. A. and Yu, B. Central limit theorems for empirical and U-processes of stationary mixing sequences. Journal of Theoretical Probability. Vol. 7 No.1, 1994, 47-71.

[4] Doukhan, P., Massart, P. and Rio, E. Invariance principles for absolutely regular empirical processes. Ann. Inst. H. Poincaré Probab. Statist. 31, no. 2, 1995, 393-427.

[5] Dudley, R. M. Universal Donsker classes and metric entropy. Ann. Probab. 15, 1987, 1306-1326.

[6] Kolčinskiĭ, V. I. On the central limit theorem for empirical measures. Theory Probab. Math. Statist. 24, 1982, 71-82,

[7] Levental, S. Uniform limit theorems for Harris recurrent Markov chains. Probability Theory and Related Fields 80, 1988, 101-118.

[8] Levental, S. Uniform CLT for Markov chain with a countable state space. Stochastic Processes and their Application 34, 1990, 245-253.

[9] Pollard, D. A central limit theorem for empirical processes. Journal of the Australian Mathematical Society (Series A) 33, 1982, 235-248.