Chapter 3

Uniform LILI for Markov chains with a countable state space

3.1 Introduction

Let $(S, \mathcal{G}, P)$ be a probability space and let $\mathcal{F}$ be a set of measurable functions on S with an envelope function F finite everywhere. Let $X_1, X_2, ...$ be a strictly stationary sequence of random variables with distribution P. We say the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if there exists a compact set K in $l^\infty(\mathcal{F})$ such that, with probability one,

$$\{(2n \log\log n)^{-\frac{1}{2}} \sum_{j=1}^{n} (f(X_j) - Ef(X_1)) : f \in \mathcal{F}\}_{n=1}^{\infty}$$

is relatively compact and its limit set is K, and the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if, with probability one,

$$\sup_n \sup_{f\in\mathcal{F}} (2n \log\log n)^{-\frac{1}{2}} \left| \sum_{j=1}^{n} (f(X_j) - Ef(X_1)) \right| < \infty.$$

Let $\{X_i\}_{i\geq 0}$ be a positive recurrent irreducible Markov chain taking values in $S = \{1, 2, 3, \cdots\}$ with the unique invariant probability measure π, N_i be the i-th hitting time of state 1,

$$m_{i,j} = E\left(\min\{n : n \geq 1, X_n = j\} \mid X_0 = i\right).$$

Recall in chapter 2 we have the uniform CLT for $\mathcal{F} = \{f : |f| \leq F\}$ under conditions $E(N_2 - N_1)^2 < \infty$ and the envelope function F of $\mathcal{F}$ satisfying $E(\sum_{N_1<j\leq N_2} F(X_j))^2 < \infty$ and

$$\sum_{k=1}^{\infty} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} < \infty \tag{3.1}$$

In this chapter we will prove the compact LIL and bounded LIL for Markov chains under a weaker condition than (3.1). Let F also be a non-negative function on S and $\mathcal{F} = \{f : |f| \leq F\}$. Assume $E(N_2 - N_1)^2 < \infty$, $E(\sum_{N_1<j\leq N_2} F(X_j))^2 < \infty$ and choose a suitable order of S as indicated in Theorem 3.1 below. If

$$(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} \to 0, \tag{3.2}$$

then the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$. Conversely, if the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ and there are $c, \alpha > 0$ such that $\pi(k) \geq ck^{-\alpha}$ for all $k \in S$, then (3.2) holds. We have the bounded LIL when (3.2) is replaced by

$$\sup_n (\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} < \infty.$$

To compare our results to those obtained from weakly dependent observations, we let $\{X_i\}$ be a strictly stationary sequence of random variables and recall the definition of the absolutely regular mixing coefficient:

$$\beta_k = \tfrac{1}{2}\sup\{\sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)| : \{A_i\}_{i=1}^{I} \text{is a partition in } \sigma_1^l \text{ and } \{B_j\}_{j=1}^{J} \text{is a partition in } \sigma_{l+k}^{\infty}, l \geq 1\}.$$

Arcones (1995) proved a compact LIL over classes of functions which satisfy a bracketing condition with respect to a norm $\|\cdot\|_{2,\beta}$ defined in [**?**]. He also obtained a compact LIL on a discrete space over classes whose envelope functions satisfy

$$\sum_{x\in S} F(x)(P(X_1 = x))^{\frac{1}{p}} < \infty,$$

where p depends on the rate of decay of mixing coefficient β_k.

We will present an example to illustrate that, in the Markov chain case, applying Arcones' empirical processes result will not get the results obtained by our approach. We also obtain a result which is better than Arcones' discrete space result in some special cases.

3.2 Empirical law of iterated logarithm

Let $\{X_j\}_{j\geq 0}$ be a positive recurrent irreducible Markov chain taking values in $S = \{1, 2, 3, \cdots\}$ with an invariant probability measure π. Let N_i be the i-th hitting time of state 1.

We define for every $f \in L^1(S, \pi)$

$$S_n(f) = \sum_{j=1}^{n} (f(X_j) - \pi(f))$$

and

$$Z_k(f) = \sum_{N_k < j \leq N_{k+1}} (f(X_j) - \pi(f))$$

for all $k \geq 1$. Then the $Z_k(f)$ are i.i.d.. If $\mathcal{F}$ is a subset of $L^2(S, \pi)$ with the induced topology, then $Z_k(\cdot)$ are almost surely continuous on $\mathcal{F}$. Denote by $\mathcal{C}(\mathcal{F},\mathbf{R})$ the set of all continuous functions from $\mathcal{F}$ to the real numbers. If $\mathcal{F}$ is a compact subset of $L^2(S, \pi)$, then $\mathcal{C}(\mathcal{F},\mathbf{R})$ equipped with supremum norm become a separable Banach space. Then we can consider $Z_k(\cdot)$ as random elements in $\mathcal{C}(\mathcal{F},\mathbf{R})$ and apply those limit theorems in separable Banach spaces.

Let (M, d) be a metric space, $\{x_n\}$ a sequence of points in M and $A \subseteq M$. We use the notation $\{x_n\} \to\to A$ if both $\lim_n d(x_n, A) = 0$ and the cluster set of $\{x_n\}$ is A. We also let $a_n = (2n \log\log n)^{\frac{1}{2}}$.

Let $\mathcal{F}$ be a compact subset of $L^2(S, \pi)$. We define $H_{L(Z_1)}$ in $\mathcal{C}(\mathcal{F},\mathbf{R})$ by canonical construction (Kuelbs 1976), and let K be the unit ball of $H_{L(Z_1)}$. The compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ means that

$$\left\{ \frac{1}{a_n} S_n(f) : f \in \mathcal{F} \right\} \to\to \frac{K}{\sqrt{m_{1,1}}} \quad \text{a.s.}$$

and the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ means that

$$\sup_n \sup_{f \in \mathcal{F}} \frac{1}{a_n} |S_n(f)| < \infty \quad \text{a.s..}$$

Comparing Theorem 2.7 of Chapter 2 and Theorem 3.1 below we see that we can obtain the compact LIL and bounded LIL under conditions which are weaker than those for the uniform CLT.

Theorem 3.1. Let F be a non-negative function on S and $\mathcal{F} = \{f : |f| \le F\}$. Suppose $E(N_2 - N_1)^2 < \infty$,

$$E\left(\sum_{N_1 < j \le N_2} F(X_j)\right)^2 < \infty, \tag{3.3}$$

and let θ_F be any bijection from S to itself such that the sequence

$$b_{\theta_F,k} = F(\theta_F(k))\pi(\theta_F(k))m_{1,\theta_F(k)}^{\frac{1}{2}}$$

is decreasing.
(I) If

$$\lim_{n\to\infty} (\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} b_{\theta_F,k} = 0 \tag{3.4}$$

then the compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$. Conversely, if the compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$, then we have (3.4) for those bijections on S such that for some $c, \alpha > 0$

$$\pi(\theta(k)) \ge ck^{-\alpha} \quad \text{for all } k \in S. \tag{3.5}$$

(II) If

$$\sup_{n} (\log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} b_{\theta_F,k} < \infty \tag{3.6}$$

then the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$. Conversely, if the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$, then we have (3.6) for those bijections on S such that (3.5) holds for some $c, \alpha > 0$.

Remarks.
(i) In (3.4), since $b_{\theta_F,k}$ is decreasing

$$\sum_{k=1}^{n} b_{\theta_F,k} \ge \sum_{k=1}^{n} b_{\theta,k}$$

for any bijection θ, and hence (3.4) is the strongest condition of all bijections. There are examples such that $(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^n b_{\theta_F,k}$ diverges but there exists θ such that $(\log\log n)^{-\frac{1}{2}} \sum_{k=1}^n b_{\theta,k} \to 0$.

(ii) The condition (3.3) does not depend on the state which we choose to use to define the hitting times N_i. (page 84, Theorem 4 in Chung).

(iii) The conditions $E(N_2-N_1)^2 < \infty$ and (3.3) are both necessary conditions for the uniform LIL over $\mathcal{F}$. (proof is after the proof of Theorem 3.1) Hence $F \in L^2(S,\pi)$ is also necessary by the remark following Theorem 2.2 in chapter 2.

(iv) We can replace (3.5) by the following condition: there are $c, \alpha > 0$ and $\Lambda \subseteq S$ such that

$$\pi(\theta(k)) \geq ck^{-\alpha} \quad \text{for all } k \in \Lambda$$

and

$$(\log\log n)^{-\frac{1}{2}} \sum_{1\leq k\leq n, k\notin\Lambda} F(\theta(k))\pi(\theta(k))m_{1,\theta(k)}^{\frac{1}{2}} \to 0 \quad \text{or} \quad < \infty.$$

Proof of Theorem 3.1. (To simplify notation, we omit θ_F and write k for $\theta_F(k)$.)

Lemma 3.2. Let F be a non-negative function on S and $\mathcal{F} = \{f : |f| \leq F\}$. Suppose $E(N_2 - N_1)^2 < \infty$,

$$E\left(\sum_{N_1<j\leq N_2} F(X_j)\right)^2 < \infty.$$

Then the compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ is equivalent to

$$\frac{1}{a_n}\left\|\sum_{j=1}^n Z_j\right\| \to 0 \quad \text{in probability,} \tag{3.7}$$

and the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ is equivalent to

$$\frac{1}{a_n}\sum_{j=1}^n Z_j \quad \text{is bounded in probability} \tag{3.8}$$

Proof. Let

$$R_n(f) = \sum_{1\le j\le N_1 \text{ or } N_{l(n)}<j\le n} \left(f(X_i) - \pi(f)\right),$$

and $l(n) = \max\{i : N_i \le n\}$, then

$$S_n(f) = \sum_{i=1}^{l(n)-1} Z_i(f) + R_n(f). \tag{3.9}$$

We have

$$\lim_{n\to\infty} \sup_{f\in\mathcal{F}} \frac{|R_n(f)|}{\sqrt{n}} = 0 \text{ a.s.}, \tag{3.10}$$

from Chung's proof (Theorem 5 page 106), since

$$\sup_{f\in\mathcal{F}} |R_n(f)| \le \sum_{1\le j\le N_1 \text{ or } N_{l(n)}<j\le n} F(X_j) + (N_1 + n - N_{l(n)})\pi(F).$$

The compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ means

$$\left\{\frac{1}{a_n} S_n(f)\right\}_{f\in\mathcal{F}} \to\to \frac{K}{\sqrt{m_{1,1}}} \quad \text{a.s..}$$

By (3.9) and (3.10) this is equivalent to

$$\left\{\frac{1}{a_n} \sum_{j=1}^{l(n)-1} Z_j(f)\right\}_{f\in\mathcal{F}} \to\to \frac{K}{\sqrt{m_{1,1}}} \quad \text{a.s..}$$

Since $\frac{a_n}{a_{l(n)-1}} \to \sqrt{m_{1,1}}$, where $a_{l(n)-1} = [2(l(n)-1)\log\log(l(n)-1)]^{\frac{1}{2}}$, the last expression is equivalent to

$$\left\{\frac{1}{a_n} \frac{a_n}{a_{l(n)-1}} \sum_{j=1}^{l(n)-1} Z_j(f)\right\}_{f\in\mathcal{F}} \to\to K \quad \text{a.s.},$$

and then

$$\left\{\frac{1}{a_n} \sum_{j=1}^{n} Z_j(f)\right\}_{f\in\mathcal{F}} \to\to K \quad \text{a.s.}, \tag{3.11}$$

since $\{l(n) : n = 1, 2, ...\} = \{1, 2, ...\}$. We consider Z_k as random vectors in $\mathcal{C}(\mathcal{F},\mathbf{R})$ and have

$$\begin{aligned} E\|Z_1\|^2 &= E\left[\left(\sup_{f\in\mathcal{F}}\left|\sum_{N_1<j\le N_2} f(X_i) - \pi(f)(N_2-N_1)\right|\right)^2\right] \\ &\le 2E\left[\sup_{f\in\mathcal{F}}\left(\left(\sum_{N_1<j\le N_2} f(X_j)\right)^2 + \pi^2(f)(N_2-N_1)^2\right)\right] \\ &\le 2\left[E\left(\sum_{N_1<j\le N_2} F(X_i)\right)^2 + \pi^2(F)E(N_2-N_1)^2\right]. \end{aligned}$$

Thus $E\|Z_1\|^2 < \infty$. Note that $EZ_1 = 0$. By applying Theorem 4.1 in Kuelbs (1977) we have that (3.11) is equivalent to

$$\frac{1}{a_n}\left\|\sum_{j=1}^n Z_j\right\| \to 0 \quad \text{in probability.}$$

Thus the compact LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ is equivalent to (3.7). Using similar arguments we can obtain that the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_j\}$ is equivalent to

$$\sup_n \frac{1}{a_n}\left\|\sum_{j=1}^n Z_j\right\| < \infty \quad \text{a.s.,} \tag{3.12}$$

and (3.12) is equivalent to

$$\frac{1}{a_n}\sum_{j=1}^n Z_j \quad \text{is bounded in probability}$$

by Theorem 4.2 in Kuelbs (1977).

Sufficient part of the compact LIL

To prove (3.4) is a sufficient condition for the compact LIL we assume (3.4) and show (3.7). First we have

$$\left\|\sum_{j=1}^n Z_j\right\| = \sup_{f\in\mathcal{F}}\left|\sum_{j=1}^n Z_j(f)\right| \le \sup_{f\in\mathcal{F}}\sum_{k=1}^\infty |f(k)|\left|\sum_{j=1}^n Z_j(1_{\{k\}})\right| \le \sum_{k=1}^\infty F(k)\left|\sum_{j=1}^n Z_j(1_{\{k\}})\right|.$$

Thus, for fixed $\varepsilon > 0$,

$$P\left(\frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| > \varepsilon\right) \;\leq\; P\left(\frac{1}{a_n}\sum_{k=1}^{\infty} F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \varepsilon\right),$$

and hence

$$\begin{aligned} & P\left(\frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| > \varepsilon\right) \\ \leq\; & \underbrace{P\left(\frac{1}{a_n}\sum_{k=1}^{n^2} F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{\varepsilon}{2}\right)}_{I} \;+\; \underbrace{P\left(\frac{1}{a_n}\sum_{k=n^2+1}^{\infty} F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{\varepsilon}{2}\right)}_{II}. \end{aligned} \tag{3.13}$$

By Markov's inequality

$$I \leq \frac{\sqrt{2}}{\varepsilon}(\log\log\; n)^{-\frac{1}{2}}\sum_{k=1}^{n^2} F(k) E\left(n^{-\frac{1}{2}}\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|\right). \tag{3.14}$$

Since $Z_i(\cdot)$ are i.i.d. and centered,

$$E(n^{-\frac{1}{2}} \mid \sum_{i=1}^{n} Z_i(1_{\{k\}}) \mid) \leq (n^{-1}E \mid \sum_{i=1}^{n} Z_i(1_{\{k\}}) \mid^2)^{\frac{1}{2}} = \left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}}. \tag{3.15}$$

Denote $\omega(k) = (E(\sum_{N_1<j\leq N_2} 1_{\{k\}}(X_j))^2)^{\frac{1}{2}}$, and then by definition,

$$\begin{aligned} \left(E\left(Z_1^2(1_{\{k\}})\right)\right)^{\frac{1}{2}} &= (E(\sum_{N_1<j\leq N_2} 1_{\{k\}}(X_j) - (N_2-N_1)\pi(k))^2)^{\frac{1}{2}} \\ &\leq \omega(k) + \pi(k)\left(E(N_2-N_1)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.16}$$

¿From (3.15) and (3.16)

$$E\left(n^{-\frac{1}{2}}\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|\right) \leq \omega(k) + \pi(k)\left(E\left(N_2-N_1\right)^2\right)^{\frac{1}{2}}. \tag{3.17}$$

Since $\sum_{k=1}^{\infty} F(k)\pi(k) < \infty$ we only have to show that

$$(\log\log\; n)^{-\frac{1}{2}}\sum_{k=1}^{n^2} F(k)\omega(k) \to 0,$$

and that is equivalent to

$$(\log\log\ n)^{-\frac{1}{2}} \sum_{k=1}^{n} F(k)\omega(k) \to 0.$$

From Chung [2, p88],

$$E(\sum_{N_1<j\le N_2} 1_{\{k\}}(X_j))^2 = 2m_{1,1}\pi^2(k)\left(m_{1,k}+m_{k,1}\right) - m_{1,1}\pi(k) \quad \text{for } k \ge 1 \tag{3.18}$$

and

$$E\left(N_2-N_1\right)^2 = 2m_{1,1}\sum_{k=1}^{\infty}\pi(k)m_{k,1} - m_{1,1}. \tag{3.19}$$

Since $E\left(N_2-N_1\right)^2 < \infty$ and $m_{1,1} < \infty$, we thus have $\sum_{k=1}^{\infty}\pi(k)m_{k,1} < \infty$, and hence

$$\sum_{k=1}^{\infty} F(k)\pi(k)m_{k,1}^{\frac{1}{2}} \le (\sum_{k=1}^{\infty} F^2(k)\pi(k))^{\frac{1}{2}}(\sum_{k=1}^{\infty}\pi(k)m_{k,1})^{\frac{1}{2}} < \infty, \tag{3.20}$$

From (3.18) we have

$$\begin{aligned} &\sum_{k=1}^{n} F(k)\omega(k) \\ \le\ & \sqrt{2m_{1,1}}\left[\sum_{k=1}^{n} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} + \sum_{k=1}^{n} F(k)\pi(k)m_{k,1}^{\frac{1}{2}}\right], \end{aligned} \tag{3.21}$$

and thus obtain the convergence by (3.20) and (3.4). For the other part we have

$$II \le \frac{2}{\varepsilon a_n}\sum_{k=n^2+1}^{\infty} F(k)E\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| \le \frac{\sqrt{8}m_{1,1}}{\varepsilon}\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\sum_{k=n^2+1}^{\infty} F(k)\pi(k), \tag{3.22}$$

since $E\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| \le nE\left|Z_1(1_{\{k\}})\right| \le 2m_{1,1}n\pi(k)$. Now we have to show that

$$\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\sum_{k=n^2+1}^{\infty} F(k)\pi(k) \to 0. \tag{3.23}$$

From (3.4)

$$(\log\log n)^{-\frac{1}{2}}\sum_{k=1}^{n} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} \to 0.$$

Write $b_k = F(k)\pi(k)m_{1,k}^{\frac{1}{2}}$. Thus for n large enough

$$\sum_{k=1}^{n} b_k \le (\log\log n)^{\frac{1}{2}}$$

and since b_k is decreasing

$$b_n \le \frac{1}{n}\sum_{k=1}^{n} b_k \le \left(\frac{\log\log n}{n^2}\right)^{\frac{1}{2}}.$$

Thus for n large enough

$$\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\left(\sum_{k=n^2+1}^{\infty} b_k^2\right)^{\frac{1}{2}} \le \left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\left(\sum_{k=n^2+1}^{\infty}\frac{\log\log k}{k^2}\right)^{\frac{1}{2}}, \quad (3.24)$$

and the right hand side converges to zero as $n \to \infty$. Since $m_{1,k} + m_{k,1} \ge m_{k,k} = (\pi(k))^{-1}$ for all k and $\sum_{k=1}^{\infty} F^2(k)\pi^2(k)m_{k,1} = M < \infty$ by (3.20)

$$\begin{aligned}\sum_{k=n^2+1}^{\infty} F(k)\pi(k) &\le \left(\sum_{k=n^2+1}^{\infty} F^2(k)\pi(k)\right)^{\frac{1}{2}}\left(\sum_{k=n^2+1}^{\infty}\pi(k)\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=n^2+1}^{\infty} F^2(k)\pi^2(k)m_{k,1} + \sum_{k=n^2+1}^{\infty} b_k^2\right)^{\frac{1}{2}},\end{aligned}$$

and hence we have (3.23) by (3.24).

Sufficient part of the bounded LIL

For the bounded LIL we assume (3.6). Hence there is a finite $M_0 > 0$ such that

$$\sup_n (\log\log n)^{-\frac{1}{2}}\sum_{k=1}^{n^2} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} < M_0.$$

To show (3.8) it suffices to show that for every $\eta > 0$ there exists $M > 0$ such that

$$\sup_n P\left(\frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| > M\right) < \eta.$$

Similar to the proof of (3.23), we can obtain

$$\sup_n \left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \sum_{k=n^2+1}^{\infty} F(k)\pi(k) < \infty.$$

Since $F \in L^1(S,\pi)$ and (3.20) we define

$$\begin{aligned} c_1 &= \sup_n (\log\log n)^{-\frac{1}{2}} \textstyle\sum_{k=1}^{n^2} F(k)\pi(k)\left(E\left(N_2 - N_1\right)^2\right)^{\frac{1}{2}} \\ c_2 &= \sup_n (\log\log n)^{-\frac{1}{2}} \textstyle\sum_{k=1}^{n^2} F(k)\pi(k) m_{k,1}^{\frac{1}{2}} \\ c_3 &= \sup_n \left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \textstyle\sum_{k=n^2+1}^{\infty} F(k)\pi(k). \end{aligned}$$

Fix $\eta > 0$ and take $M = \frac{1}{\eta}\left[2\sqrt{2}\left(c_1 + \left[\sqrt{2m_{1,1}}\left(M_0 + c_2\right)\right]\right) + 4\sqrt{2}m_{1,1}c_3\right]$. From (3.13)

$$\begin{aligned} & P\left(\frac{1}{a_n}\left\|\sum_{j=1}^{n} Z_j\right\| > M\right) \\ \le\ & \underbrace{P\left(\frac{1}{a_n}\sum_{k=1}^{n^2} F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{M}{2}\right)}_{I} + \underbrace{P\left(\frac{1}{a_n}\sum_{k=n^2+1}^{\infty} F(k)\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right| > \frac{M}{2}\right)}_{II}. \end{aligned}$$

From (3.14)

$$I \le \frac{\sqrt{2}}{M}(\log\log\ n)^{-\frac{1}{2}} \sum_{k=1}^{n^2} F(k) E\left(n^{-\frac{1}{2}}\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|\right).$$

From (3.17) and (3.21)

$$\sup_n (\log\log\ n)^{-\frac{1}{2}} \sum_{k=1}^{n^2} F(k) E\left(n^{-\frac{1}{2}}\left|\sum_{j=1}^{n} Z_j(1_{\{k\}})\right|\right) \le c_1 + \left[\sqrt{2m_{1,1}}\left(M_0 + c_2\right)\right],$$

thus $I \leq \frac{\sqrt{2}}{M}\left(c_1 + \left[\sqrt{2m_{1,1}}\,(M_0 + c_2)\right]\right) \leq \frac{\eta}{2}$. We also obtain that $II \leq \frac{\eta}{2}$ since from (3.22)

$$II \leq \frac{\sqrt{8}m_{1,1}}{M}\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \sum_{k=n^2+1}^{\infty} F(k)\pi(k) \leq \frac{\sqrt{8}m_{1,1}c_3}{M} \leq \frac{\eta}{2}.$$

So the sufficient part of the proof is completed.

Necessary part of LIL

For the necessary part, we suppose (3.7) holds to show (3.4), and (3.8) holds to show (3.6). We need further notation. Let

$$Y_k(f) = \sum_{N_k < j \leq N_{k+1}} f(X_j) - m_{1,1}\pi(f)$$

and

$$U_k(f) = (m_{1,1} - (N_{k+1} - N_k))\,\pi(f).$$

Then

$$Z_k(f) = Y_k(f) + U_k(f).$$

Lemma 3.3. Suppose (3.3) holds.
(a). If (3.7) holds, then

$$\frac{1}{a_n}\left\|\sum_{j=1}^{n} Y_j\right\| \to 0 \quad \text{in probability.}$$

(b). If (3.8) holds, then

$$\sup_n \frac{1}{a_n}\left\|\sum_{j=1}^{n} Y_j\right\| < \infty \quad \text{a.s..}$$

Proof. Since $Z_k = Y_k + U_k$, we only have to show that

$$\frac{1}{a_n}\left\|\sum_{j=1}^{n} U_j\right\| \to 0 \quad \text{in probability.}$$

Kolmogorov's LIL holds for the i.i.d. sequence $\{N_{j+1}-N_j\}_{j\geq 1}$ since $E(N_2-N_1)^2<\infty$, and that is equivalent to

$$\frac{1}{a_n}\left|\sum_{j=1}^{n}(m_{1,1}-(N_{j+1}-N_j))\right|\to 0 \quad \text{in probability.}$$

We have the desired result because

$$\left\|\sum_{j=1}^{n}U_j\right\|\leq\left|\sum_{j=1}^{n}(m_{1,1}-(N_{j+1}-N_j))\right|\pi(F). \tag{3.25}$$

For part (b), similarly we only have to show that

$$\sup_n\sup_{f\in\mathcal{F}}\frac{1}{a_n}\left|\sum_{j=1}^{n}U_j(f)\right|<\infty \quad \text{a.s..}$$

Since $E(N_2-N_1)^2<\infty$ we have

$$\sup_n\frac{1}{a_n}\left|\sum_{j=1}^{n}(m_{1,1}-(N_{j+1}-N_j))\right|<\infty \quad \text{a.s.,}$$

and thus by (3.25)

$$\sup_n\sup_{f\in\mathcal{F}}\frac{1}{a_n}\left|\sum_{j=1}^{n}U_j(f)\right|<\infty \quad \text{a.s..}$$

Lemma 3.4. Suppose (3.3) holds.
(a). If (3.7) holds, then

$$\frac{1}{a_n}E\left\|\sum_{j=1}^{n}Y_j\right\|\to 0.$$

(b). If (3.8) holds, then

$$\sup_n\frac{1}{a_n}E\left\|\sum_{j=1}^{n}Y_j\right\|<\infty.$$

Proof. Suppose (3.7) holds. Then we have (3.8) and

$$\sup_n \frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\| < \infty \quad \text{a.s.}$$

by part (b) of Lemma 3.3. Thus

$$E\left[\sup_n \frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\|\right] < \infty \quad \text{is equivalent to} \quad E\left[\sup_n \frac{1}{a_n} \|Y_n\|\right] < \infty.$$

(Ledoux and Talagrand, Corollary 6.12). That is, since (3.3) holds we have

$$E\left[\sup_n \frac{1}{a_n} \sum_{N_n < j \le N_{n+1}} F(X_j)\right] < \infty.$$

(Here we use that if $\{X_j\}$ is i.i.d. and real valued, then $EX_1^2 < \infty$ implies $E[\sup_n n^{-\frac{1}{2}} |X_n|] < \infty$). Since

$$\|Y_n\| \le \sum_{N_n < j \le N_{n+1}} F(X_j) + m_{1,1}\pi(F),$$

we have $E\left[\sup_n \frac{1}{a_n} \|Y_n\|\right] < \infty$, and hence $E\left[\sup_n \frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\|\right] < \infty$. From (3.7)

$$P\left(\frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\| > \varepsilon\right) \to 0,$$

and hence

$$\overline{\lim_n} E\Psi_n \le \overline{\lim_n} E\left[\Psi_n 1_{\{\Psi_n \le \varepsilon\}}\right] + \overline{\lim_n} E\left[\Psi_n 1_{\{\Psi_n > \varepsilon\}}\right] \le \varepsilon + \overline{\lim_n} E\left[\left(\sup_n \Psi_n\right) 1_{\{\Psi_n > \varepsilon\}}\right] = \varepsilon,$$

where $\Psi_n = \frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\|$. Since ε is arbitrary we have the desired result. For part (b), the above argument and (3.8) imply

$$E\left[\sup_n \frac{1}{a_n} \left\| \sum_{j=1}^n Y_j \right\|\right] < \infty,$$

and this implies $\sup_n \frac{1}{a_n} E \left\| \sum_{j=1}^n Y_j \right\| < \infty$.

Lemma 3.5. We have

$$E \left\| \sum_{j=1}^n Y_j \right\| = \sum_{k=1}^\infty F(k) E \left| \sum_{j=1}^n Y_j(1_{\{k\}}) \right| .$$

Proof. Pointwise, for all ω, we have

$$\sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n Y_j(f) \right| = \sup_{f \in \mathcal{F}} \left| \sum_{k=1}^\infty f(k) \sum_{j=1}^n Y_j(1_{\{k\}}) \right| = \sum_{k=1}^\infty F(k) \left| \sum_{j=1}^n Y_j(1_{\{k\}}) \right|$$

i.e. since $\mathcal{F} = \{f : |f(x)| \leq F(x) \text{ for all } x \in S\}$ we can pick up any sequence of positive or negative that is required.

Lemma 3.6. Suppose (3.3) holds and there is a bijection θ from S to itself such that (3.5) holds. Then there are $c', M > 0$ such that

$$n^{-\frac{1}{2}} E \left| \sum_{j=1}^n Y_j(1_{\{\theta(k)\}}) \right| \geq c' \left[E Y_1^2(1_{\{\theta(k)\}}) \right]^{\frac{1}{2}}$$

for all $k \in S$ and $n \geq \max\{k^{4\alpha}(EY_1^2(1_{\{\theta(k)\}}))^{-2}, M\}$.

Proof. (omit θ) By the Marcinkiewicz-Zygmund inequality

$$n^{-\frac{1}{2}} E \left| \sum_{j=1}^n Y_j(1_{\{k\}}) \right| \geq c_1 E \left[\left(n^{-1} \sum_{j=1}^n Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} \right]$$

where $c_1 > 0$ is a constant which is independent of the random variables. Note that

$$\begin{aligned} & E \left[\left(\frac{1}{n} \sum_{j=1}^n Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} \right] \\ \geq \;& E \left[\left(\frac{1}{n} \sum_{j=1}^n Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} 1_{\left\{ \left(n^{-1} \sum_{j=1}^n Y_j^2(1_{\{k\}}) \right)^{\frac{1}{2}} \geq \left[EY_1^2(1_{\{k\}}) \right]^{\frac{1}{2}} \right\}} \right] \\ \geq \;& \left[EY_1^2(1_{\{k\}}) \right]^{\frac{1}{2}} P \left(\frac{1}{n} \sum_{j=1}^n Y_j^2(1_{\{k\}}) \geq EY_1^2(1_{\{k\}}) \right) . \end{aligned}$$

By the Berry-Esseen theorem

$$P\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^2(1_{\{k\}}) \geq EY_1^2(1_{\{k\}})\right) \geq \frac{1}{2} - \frac{3n^{-\frac{1}{2}} EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}}.$$

Thus it is enough to show that there is a $M > 0$ such that

$$\frac{3n^{-\frac{1}{2}} EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq \frac{1}{4} \quad \text{for all } n \geq \max\{k^{4\alpha}(EY_1^2(1_{\{k\}}))^{-2}, M\}.$$

Let $T_i = \min\{n \geq 1 : X_n = i\}$, $a_k = P_1(T_k < T_1)$, $b_k = P_k(T_k < T_1)$, and $1 - b_k = P_k(T_k > T_1)$ for $k > 1$. Denote $W(k) = \sum_{N_1 < j \leq N_2} 1_{\{k\}}(X_j)$ then

$$EW^l(k) = \sum_{m=1}^{\infty} m^l P(W(k) = m) = a_k b_k^{-1}(1 - b_k) \sum_{m=1}^{\infty} m^l b_k^m$$

since $P(W(k) = m) = a_k b_k^{m-1}(1 - b_k)$. By computation

$$\begin{aligned}
\textstyle\sum_{m=1}^{\infty} m b_k^m &= \tfrac{b_k}{(1-b_k)^2} \\
\textstyle\sum_{m=1}^{\infty} m^2 b_k^m &= \tfrac{2b_k}{(1-b_k)^3} - \tfrac{b_k}{(1-b_k)^2} \\
\textstyle\sum_{m=1}^{\infty} m^4 b_k^m &= \tfrac{24b_k}{(1-b_k)^5} - \tfrac{36b_k}{(1-b_k)^4} + \tfrac{14b_k}{(1-b_k)^3} - \tfrac{b_k}{(1-b_k)^2} \\
\textstyle\sum_{m=1}^{\infty} m^6 b_k^m &= \tfrac{720b_k}{(1-b_k)^7} - \tfrac{1800b_k}{(1-b_k)^6} + \tfrac{1560b_k}{(1-b_k)^5} - \tfrac{540b_k}{(1-b_k)^4} + \tfrac{62b_k}{(1-b_k)^3} - \tfrac{b_k}{(1-b_k)^2}.
\end{aligned}$$

Thus $EW(k) = \frac{a_k}{1-b_k}$. Then

$$EY_1^2(1_{\{k\}}) = E\left(W(k) - EW(k)\right)^2 = a_k\left(\frac{2}{(1-b_k)^2} - \frac{1}{(1-b_k)}\right) - \frac{a_k^2}{(1-b_k)^2}$$

and

$$EY_1^2(1_{\{k\}}) \leq \frac{2a_k}{(1-b_k)^2}. \tag{3.26}$$

Now we claim that there is a $M_1 > 0$ such that

$$\frac{EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq M_1 a_k^{-\frac{1}{2}} \quad \text{for } k \in V, \tag{3.27}$$

where V is the set of those k such that a_k is sufficiently small. Note that $S - V$ is finite since

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} \frac{a_k}{1-b_k} = \sum_{k=1}^{\infty} EW(k) = \sum_{k=1}^{\infty} m_{1,1}\pi(k) = m_{1,1}.$$

If $b_k < \frac{1}{2}$ then there is a $M_2 > 0$ such that for all $k \in V$,

$$\begin{array}{rcl} EY_1^6(1_{\{k\}}) & \le & EW^6(k) \\ & = & a_k \left(\frac{720}{(1-b_k)^6} - \frac{1800}{(1-b_k)^5} + \frac{1560}{(1-b_k)^4} - \frac{540}{(1-b_k)^3} + \frac{62}{(1-b_k)^2} - \frac{1}{(1-b_k)} \right) \\ & \le & M_2 a_k \end{array}$$

and

$$\begin{array}{rcl} EY_1^4(1_{\{k\}}) & = & E\left(W(k) - \frac{a_k}{1-b_k} \right)^4 \\ & = & a_k \left(\frac{24}{(1-b_k)^4} - \frac{36}{(1-b_k)^3} + \frac{14}{(1-b_k)^2} - \frac{1}{(1-b_k)} \right) - a_k^4 \left(\frac{1}{(1-b_k)^4} \right) \\ & & -4a_k^2 \left(\frac{6}{(1-b_k)^4} - \frac{6}{(1-b_k)^3} + \frac{1}{(1-b_k)^2} \right) + 6a_k^3 \left(\frac{2}{(1-b_k)^4} - \frac{1}{(1-b_k)^3} \right) \\ & \ge & \frac{a_k}{4}. \end{array}$$

Thus if $b_k < \frac{1}{2}$ then

$$\frac{EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \le 8M_2 a_k^{-\frac{1}{2}} \quad \text{for } k \in V.$$

If $b_k \ge \frac{1}{2}$ then there is a $M_3 > 0$ such that $EY_1^6(1_{\{k\}}) \le M_3 \frac{a_k}{(1-b_k)^6}$ and $EY_1^4(1_{\{k\}}) \ge \frac{a_k}{4(1-b_k)^4}$ for $k \in V$. Thus if $b_k \ge \frac{1}{2}$ then

$$\frac{EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \le 8M_3 a_k^{-\frac{1}{2}} \quad \text{for } k \in V.$$

So (3.27) holds if we take $M_1 = \max\{8M_2, 8M_3\}$. We also claim that

$$1 - b_k \ge ck^{-\alpha} \quad \text{except for finite many } k. \tag{3.28}$$

Note that $E(N_2 - N_1)^2 < \infty$ implies $\sum_{k=1}^{\infty} \pi(k) m_{k,1} < \infty$ by (3.19). Thus by (3.5)

$$m_{k,1} \le c^{-1} k^{\alpha} \quad \text{except for finite many } k.$$

However, we have

$$\begin{array}{rcl} m_{k,1} & \ge & \sum_{j=1}^{\infty} j \left(P_k\left(T_1 > T_k\right)\right)^{j-1} P_k\left(T_1 < T_k\right) \\ & = & \sum_{j=1}^{\infty} j b_k^{j-1} \left(1 - b_k\right) \\ & = & \left(1 - b_k\right)^{-1}. \end{array}$$

Thus

$$1-b_k \geq m_{k,1}^{-1} \geq ck^{-\alpha} \quad \text{except for finite many } k.$$

From (3.26), if $n \geq k^{4\alpha}\left(EY_1^2(1_{\{k\}})\right)^{-2}$, then

$$\frac{1}{\sqrt{n}} \leq \frac{EY_1^2(1_{\{k\}})}{k^{2\alpha}} \leq \frac{2a_k}{k^{2\alpha}\left(1-b_k\right)^2},$$

and from (3.27) and (3.28) for $k \in V$,

$$\frac{3n^{-\frac{1}{2}}EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq 3M_1 n^{-\frac{1}{4}}\sqrt{\frac{1}{a_k\sqrt{n}}} \leq 3M_1 n^{-\frac{1}{4}}\sqrt{\frac{2}{k^{2\alpha}\left(1-b_k\right)^2}} \leq 3\sqrt{2}M_1 c^{-1} n^{-\frac{1}{4}}$$

except for finite many k. Now we can choose $M > 0$ such that for all $k \in S$

$$\frac{3n^{-\frac{1}{2}}EY_1^6(1_{\{k\}})}{\left(EY_1^4(1_{\{k\}})\right)^{\frac{3}{2}}} \leq \frac{1}{4} \quad \text{for all } n \geq \max\{M, k^{4\alpha}(EY_1^2(1_{\{k\}}))^{-2}\},$$

and this complete the proof of Lemma 3.6.

Continue the proof of necessary part.

Now back to the proof of necessary part, (omit θ) let

$$U = \left\{k : F(k)\left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} \geq k^{-\frac{3}{2}}\right\}.$$

Since $F \in L^2(S,\pi)$ and (3.5) holds,

$$F(k) \leq c^{-\frac{1}{2}}k^{\frac{\alpha}{2}} \quad \text{except for finite many } k.$$

Thus for $k \in U$

$$EY_1^2(1_{\{k\}}) \geq ck^{-3-\alpha}$$

and

$$k^{4\alpha}\left(EY_1^2(1_{\{k\}})\right)^{-2} \leq c^{-2}k^{6\alpha+6}$$

except for finite many k. We denote the exceptional set by V' and then the conclusion of Lemma 3.6 can be replaced by that for $k \in U - V'$, there are c' and M such that

$$n^{-\frac{1}{2}}E\left|\sum_{j=1}^{n} Y_j(1_{\{k\}})\right| \geq c'\left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} \quad \text{for all } n \geq \max\{M, c^{-2}k^{6\alpha+6}\}.$$

Thus for $n \geq M$

$$n^{-\frac{1}{2}} \sum_{k \in \Lambda_n} F(k) E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| \geq \sum_{k \in \Lambda_n} F(k) c' \left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \tag{3.29}$$

where $\Lambda_n = \{k : k \in U - V' \text{ and } 1 \leq k \leq (c^2 n)^{\frac{1}{6\alpha+6}}\}$. For the compact LIL, from part (a) of Lemma 3.4 and Lemma 3.5 we have

$$\frac{1}{a_n} \sum_{k=1}^{\infty} F(k) E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| \to 0,$$

thus by (3.29)

$$(\log \log n)^{-\frac{1}{2}} \sum_{k \in \Lambda_n} F(k) \left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \to 0.$$

Since $\sum_{k \notin U} F(k)(EY_1^2(1_{\{k\}}))^{\frac{1}{2}} \leq \sum_{k \notin U} k^{-\frac{3}{2}} < \infty$ and V' is finite

$$(\log \log n)^{-\frac{1}{2}} \sum_{1 \leq k \leq (c^2 n)^{\frac{1}{6\alpha+6}}} F(k) \left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \to 0.$$

Let $m = (c^2 n)^{\frac{1}{6\alpha+6}}$ then $n = c^{-2} m^{6\alpha+6}$ and hence

$$\left(\log \log \left(c^{-2} m^{6\alpha+6} \right) \right)^{-\frac{1}{2}} \sum_{1 \leq k \leq m} F(k) \left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \to 0.$$

Thus

$$(\log \log m)^{-\frac{1}{2}} \sum_{1 \leq k \leq m} F(k) \left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \to 0.$$

Note that from (3.18)

$$\left(EY_1^2(1_{\{k\}}) \right)^{\frac{1}{2}} \geq \left(E \left(\sum_{N_1 < j \leq N_2} 1_{\{k\}}(X_j) \right)^2 \right)^{\frac{1}{2}} - m_{1,1}\pi(k) \geq \sqrt{m_{1,1}}\,\pi(k) m_{1,k}^{\frac{1}{2}} - 2 m_{1,1} \pi(k),$$

thus

$$(\log \log m)^{-\frac{1}{2}} \sum_{1 \leq k \leq m} F(k) \pi(k) m_{1,k}^{\frac{1}{2}} \to 0$$

since $\sum_{1\le k\le\infty} F(k)\pi(k) < \infty$.

For the bounded LIL, from part (b) of Lemma 3.4 and Lemma 3.5 we have

$$\sup_n \frac{1}{a_n} \sum_{k=1}^{\infty} F(k) E \left| \sum_{j=1}^{n} Y_j(1_{\{k\}}) \right| < \infty,$$

thus by (3.29)

$$\sup_n (\log\log n)^{-\frac{1}{2}} \sum_{k\in\Lambda_n} F(k) \left(EY_1^2(1_{\{k\}})\right)^{\frac{1}{2}} < \infty.$$

Using a similar argument we can obtain

$$\sup_m (\log\log m)^{-\frac{1}{2}} \sum_{1\le k\le m} F(k)\pi(k) m_{1,k}^{\frac{1}{2}} < \infty.$$

Proof of Remark 3 following Theorem 3.1. We want to show that $E(N_2 - N_1)^2 < \infty$ and

$$E\left(\sum_{N_1<j\le N_2} F(X_j) \right)^2 < \infty$$

are necessary conditions for the LIL over $\mathcal{F} = \{f : |f| \le F\}$. Suppose the LIL holds for $\mathcal{F} = \{f : |f| \le F\}$ and F is strictly positive at least at a point of S. We need the following lemma from Chen ((2.16) of Theorem 2.2 of Chapter 3).

Lemma. If the LIL holds for F, i.e.

$$\limsup_{n\to\infty} \frac{|S_n(F)|}{a_n} < \infty \quad \text{a.s.}$$

then

$$\limsup_{n\to\infty} \frac{1}{a_n} \left| \sum_{j=N_{l(n)}+1}^{n} (F(X_j) - \pi(F)) \right| = 0 \quad \text{a.s.}$$

where $l(n) = \max\{k : N_k \le n\}$.

Recall that

$$S_n(F) = \sum_{j=1}^{N_1}(F(X_j) - \pi(F)) + \sum_{k=1}^{l(n)-1} Z_k(F) + \sum_{j=N_{l(n)}+1}^{n} (F(X_j) - \pi(F)).$$

Since the first term of the right hand side is finite a.s. and does not depend on n,

$$\limsup_{n\to\infty} \frac{1}{a_n}\left|\sum_{k=1}^{l(n)-1} Z_k(F)\right| < \infty \quad \text{a.s..}$$

Thus since $l(n)/n \to 1/\mu$ a.s.,

$$\limsup_{n\to\infty} \frac{1}{a_n}\left|\sum_{k=1}^{n} Z_k(F)\right| < \infty \quad \text{a.s.,} \tag{3.30}$$

and (3.30) is equivalent to $E(Z_1(F))^2 < \infty$, because the LIL is equivalent to the second moment finite for real valued i.i.d. random variables. Assuming $F(1) > 0$ to be specific, we thus have the LIL for $1_{\{1\}}$ implies $E(Z_1(1_{\{1\}}))^2 = E(1 - \pi(1)(N_2 - N_1))^2 < \infty$ and hence $E(N_2 - N_1)^2 < \infty$. Thus

$$\left[E\left(\sum_{N_1<j\le N_2} F(X_j)\right)^2\right]^{\frac{1}{2}} \le \left[E(Z_1(F))^2\right]^{\frac{1}{2}} + \pi(F)\left[E(N_2 - N_1)^2\right]^{\frac{1}{2}} < \infty.$$

3.3 Comparison with mixing results

It is known that a positive recurrent irreducible Markov chain has convergent absolutely regular mixing coefficients [**?**]. Using empirical process LIL's for stationary sequences satisfying absolutely regular mixing conditions one can also obtain results similar to those above. However, our conditions are less restrictive than those required for a mixing process application to these problems.

Let $X_1, X_2, \ldots$ be a strictly stationary sequence of random variables with distribution P, and assume that the absolutely regular mixing coefficient sequence $\{\beta_k\}$ satisfies the summability condition $\sum_{k\ge1}\beta_k < \infty$. Define the mixing rate function $\beta(\cdot)$ by $\beta(t) = \beta_{[t]}$ if $t \ge 1$, and $\beta(t) = 1$ otherwise. For any numerical function f, we denote by Q_f the quantile function of $|f(X_1)|$, that is

$$Q_f(u) = \inf\{t : P(|f(X_1)| > t) \le u\}.$$

Let $\mathcal{F}$ be a class of functions in the function space $L_{2,\beta}(P)$, here the norm is defined by

$$\|f\|_{2,\beta} = \left(\int_0^1 \beta^{-1}(u)\,(Q_f(u))^2\,du\right)^{\frac{1}{2}},$$

where $\beta^{-1}(u) = \inf\{t : \beta(t) \le u\}$.Let V is a subspace of the space of measurable functions on S such that $\mathcal{F} \subseteq V$, and let $\|\cdot\|$ be a norm on V. Define the bracketing number of $\mathcal{F}$ with respect to norm $\|\cdot\|$ and V by letting, for $\varepsilon > 0$, $N_{[\,]}(\varepsilon, \mathcal{F}, \|\cdot\|)$ be the minimal number of brackets $[g_1, h_1], ..., [g_n, h_n]$, with all $g_i, h_i \in V$, such that for all $f \in \mathcal{F}$ there exists $[g_i, h_i]$, for some $i, 1 \le i \le n$ with $g_i \le f \le h_i$ and $\|h_i - g_i\| < \varepsilon$.

Doukhan, Massart and Rio (1995) proved that a sufficient condition for the uniform CLT holding over $\mathcal{F}$ is that

$$\int_0^1 \left(\log N_{[\,]}(\varepsilon, \mathcal{F}, \|\cdot\|_{2,\beta})\right)^{\frac{1}{2}} d\varepsilon < \infty \tag{3.31}$$

where $N_{[\,]}(\varepsilon, \mathcal{F}, \|\cdot\|_{2,\beta})$ is the bracketing number of $\mathcal{F}$ with respect to the norm $\|\cdot\|_{2,\beta}$ and $L_{2,\beta}(P)$.

Arcones (1995) [1, Theorem 5 and Theorem 9] obtained a compact LIL over $\mathcal{F}$ with respect to $\{X_i\}$, under the following conditions:
(i) $F(X_1) \in L_p$,
(ii) $\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty$,
(iii) (3.31) holds,
where $p > 2$ and $F(x) = \sup_{f\in\mathcal{F}} |f(x)|$. Furthermore, in case the random variables take values in a discrete space S, then the compact LIL holds over $\mathcal{F}$ under the following conditions:
(i) $\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty$, and
(ii) the envelope function F satisfies $\sum_{k\in S} F(k)(P(X_1 = k))^{\frac{1}{p}} < \infty$.

Since the uniform CLT implies the LIL, we only need examples such that the conditions of uniform CLT hold, i.e. $E(N_2 - N_1)^2 < \infty$,

$$E\left(\sum_{N_1 < j \le N_2} F(X_j)\right)^2 < \infty \tag{3.32}$$

and

$$\sum_{k=1}^{\infty} F(k)\pi(k)m_{1,k}^{\frac{1}{2}} < \infty, \tag{3.33}$$

but Arcones' conditions fail.

Example 3.7. Let $\{X_i\}$ be a stationary Markov chain with transition probability

$$P_{n,n+1} = (\frac{n}{n+1})^s,\ P_{n,1} = 1 - P_{n,n+1} \quad \text{for all } n \geq 1 \text{ and some } s > 1.$$

Let $F(k) = k^t$ for some $t \geq 0$. We have showed in the example 2.9 of chapter 2 that in the case $t = 0$ and $2 < s < 5/2$, (3.32) and (3.33) hold, but the breaketing condition (3.31) fails.

We need that (3.32) and (3.33) hold, but the conditions $\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty$ and $\sum_{k\in S} F(k)(P(X_1 = k))^{\frac{1}{p}} < \infty$ not both hold. Recall $\beta_k \gg k^{1-s}$ and by computation (3.32) and (3.33) holds if $s > 2t + 2$, but

$$\sum_{k=1}^{\infty} F(k)(\pi(k))^{\frac{1}{2+\delta}} < \infty \quad \text{only if} \quad \delta \leq \frac{s - 2t - 2}{t + 1}.$$

Take $s = 4.2$ and $t = 1$. If $\sum_{k\in S} F(k)(P(X_1 = k))^{\frac{1}{p}} < \infty$, then $p - 2 \leq \frac{1}{10}$, and then

$$\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k \gg \sum_{k=1}^{\infty} k^{-3.2} k^{20} \log\log k = \infty.$$

We also have a little improvement of Arcones' LIL for discrete spaces in the special case $\mathcal{F} = \{1_A : A \subseteq S\}$.

Proposition 3.8. Suppose $\beta_k = O(k^{-a})$, for some $a > 1$. If

$$\sum_{k=1}^{\infty} p_k^{\frac{a-1}{2a}} < \infty, \tag{3.34}$$

then (3.31) (with respect to $\mathcal{F} = \{1_A : A \subseteq S\}$) holds. Furthermore, if $p_k \approx k^{-b}$ and

$$\sum_{k=1}^{\infty} \left(k^{-b}\right)^{\frac{1}{2}\left(\frac{a-1}{a} + \frac{1}{ab}\right)} < \infty, \tag{3.35}$$

then (3.31) also holds.

Remark. In case $p_k \approx k^{-b}$, (3.34) is equivalent to $b > \frac{2a}{a-1}$, but (3.35) is equivalent to $b > \frac{2a-1}{a-1}$. Thus condition (3.35) is weaker than condition (3.34) when $p_k \approx k^{-b}$.

Proof of Proposition 3.8. (adapted from Dudley's proof). We can assume $p_k \geq p_l$ for $k \leq l$. Let r_j be the number of values of k such that

$$4^{-j-1} < p_k^{\frac{a-1}{2a}} \leq 4^{-j} \quad j = 0, 1, 2, ... \tag{3.36}$$

and let $c_j = r_j 4^{-j}$, then $\sum_{j=1}^{\infty} c_j < \infty$. For $n \geq n_0$ large enough there is a unique $j(n)$ such that

$$\sum_{j > j(n)} \frac{c_j}{4^j} \leq 4^{-n} < \sum_{j \geq j(n)} \frac{c_j}{4^j}.$$

Let $k_n = \sum_{j=1}^{j(n)} r_j$, and $A = \{k : k > k_n\}$. Suppose $\beta_k \leq ck^{-a}$ for some constant $c > 0$. Then $\beta^{-1}(u) \leq (c^{\frac{1}{a}} + 1)u^{-\frac{1}{a}}$,

$$\|1_A\|_{2,\beta} \leq \left(\frac{\left(c^{\frac{1}{a}} + 1\right) a}{a-1}\right)^{\frac{1}{2}} \left(\left(\sum_{k > k_n} p_k\right)^{\frac{a-1}{a}}\right)^{\frac{1}{2}},$$

and

$$\left(\sum_{k > k_n} p_k\right)^{\frac{a-1}{a}} \leq \sum_{k > k_n} p_k^{\frac{a-1}{a}} \leq \sum_{j > j(n)} r_j 4^{-2j} \leq 4^{-n}.$$

Thus

$$\|1_A\|_{2,\beta} \leq c' 2^{-n},$$

where $c' = [(c^{\frac{1}{a}} + 1)a(a-1)^{-1}]^{\frac{1}{2}}$. Let A_i run over all subsets of $\{1, 2, ..., k_n\}$ where $i = 1, ..., 2^{k_n}$. Let $B_i = A_i \cup A$. Then for any $C \subseteq S$, let $A_i = C \cap \{1, 2, ..., k_n\}$. Then $1_{A_i} \leq 1_C \leq 1_{B_i}$ and

$$\|1_{B_i} - 1_{A_i}\|_{2,\beta} = \|1_A\|_{2,\beta} \leq c' 2^{-n}.$$

Thus $N_{[\,]}(c' 2^{-n}, \mathcal{F}, \|\cdot\|_{2,\beta}) \leq 2^{k_n}$. We observe that (3.31) is equivalent to

$$\sum_{n=1}^{\infty} 2^{-n} \left(\log N_{[\,]}(c' 2^{-n}, \mathcal{F}, \|\cdot\|_{2,\beta})\right)^{\frac{1}{2}} < \infty,$$

thus it will be enough to prove

$$\sum_{n=1}^{\infty} k_n^{\frac{1}{2}} 2^{-n} < \infty,$$

and this follows exactly from Dudley's argument.

For the special case $c_1 k^{-b} \leq p_k \leq c_2 k^{-b}$ for all k, we have

$$\left(\sum_{k>k_n} p_k\right)^{\frac{a-1}{a}} \leq \left(\int_{k_n}^{\infty} c_2 x^{-b} dx\right)^{\frac{a-1}{a}} = \left(c_2 \left(b-1\right)\right)^{\frac{a-1}{a}} k_n^{(1-b)\frac{a-1}{a}},$$

and

$$\sum_{k>k_n} p_k^{\frac{a-1}{a}+\frac{1}{ab}} \geq \int_{k_n+1}^{\infty} c_1 x^{-b\left(\frac{a-1}{a}+\frac{1}{ab}\right)} dx = \frac{c_1 a}{\left(b-1\right)\left(a-1\right)} \left(k_n+1\right)^{(1-b)\frac{a-1}{a}}.$$

Thus

$$\left(\sum_{k>k_n} p_k\right)^{\frac{a-1}{a}} \leq M \sum_{k>k_n} p_k^{\frac{a-1}{a}+\frac{1}{ab}},$$

where M is a constant depending on $c_{1,} c_2, a$ and b. The previous argument works if we modify (3.36) by

$$4^{-j-1} < p_k^{\frac{1}{2}\left(\frac{a-1}{a}+\frac{1}{ab}\right)} \leq 4^{-j} \quad j = 0, 1, 2, ...$$

Let $\beta_k \approx k^{-a}$ for some $a > 1$. Then we combine Proposition 3.8 and Theorem 5 in [1] for general state spaces to obtain that

$$\sum_{k=1}^{\infty} F(k)(P(X_1 = k))^{\frac{1}{p}} < \infty \tag{3.37}$$

for $p = \frac{2a}{a-1}$ is a sufficient condition for the compact LIL over $\mathcal{F} = \{1_A : A \subseteq S\}$. However, since Arcones' condition $\sum_{k=1}^{\infty} \beta_k k^{\frac{2}{p-2}} \log\log k < \infty$ is equivalent to $p > \frac{2a}{a-1}$, Arcones' theorem for discrete state spaces requires that (3.37) holds for some $p > \frac{2a}{a-1}$. In the case $P(X_1 = k) \approx k^{-b}$, our condition is $b > \frac{2a-1}{a-1}$ and is less restrictive than Arcones' condition $b > \frac{2a}{a-1}$.

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