Chapter 5

Uniform LIL for Markov chains with a general state space

5.1 Introduction

Let $(S, \mathcal{G}, P)$ be a probability space and let $\mathcal{F}$ be a set of measurable functions on S with an envelope function F finite everywhere. Let $X_1, X_2, ...$ be a strictly stationary sequence of random variables with distribution P. We say the compact LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if there exists a compact set K in $l^\infty(\mathcal{F})$ such that, with probability one,

$$\{(2n \log\log n)^{-\frac{1}{2}} \sum_{j=1}^{n} (f(X_j) - Ef(X_1)) : f \in \mathcal{F}\}_{n=1}^{\infty}$$

is relatively compact and its limit set is K, and the bounded LIL holds over $\mathcal{F}$ with respect to $\{X_i\}$ if, with probability one,

$$\sup_n \sup_{f \in \mathcal{F}} (2n \log\log n)^{-\frac{1}{2}} \left| \sum_{j=1}^{n} (f(X_j) - Ef(X_1)) \right| < \infty.$$

Alexander and Talagrand [1] proved compact and bounded LIL's on VC classes of functions in the i.i.d. case. Let $\mathcal{F}$ be a countably determined VC graph class of functions on $(S, \mathcal{G}, P)$ with envelope function F satisfying

$$E\left(\frac{F^2(X)}{LLF(X)}\right) < \infty.$$

If

$$\sup_{f\in\mathcal{F}} Var\ f(X_1) < \infty$$

then the bounded LIL holds over $\mathcal{F}$. If

$$\mathcal{F} \text{ is } \rho\text{-totally bounded,}$$

where $\rho(f,g) = \|(f-g) - P(f-g)\|_2$, then the compact LIL holds over $\mathcal{F}$.

We will extend the results to regenerative processes and Markov chains. What we prove is as follows. Let $\{X_n\}$ be a regenerative process with renewal times N_i, taking values in S. Let Q be a measure on S, $N_2(\varepsilon, \mathcal{F}, Q)$ be the minimum m for which there exists $g_1, ..., g_m$ in $L^2(Q)$ such that, for all $f \in \mathcal{F}$, $\| f - g_i \|_{L^2(Q)} < \varepsilon$, for some $1 \le i \le m$, and

$$N_2(\varepsilon, \mathcal{F}) = \sup_Q (N_2(\varepsilon, \mathcal{F}, Q))$$

where the sup is taken on all the measures on S with finite support. Suppose $E(N_2 - N_1)^2 < \infty$. Let $\mathcal{F}$ be a countable family of functions on S satisfying

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty$$

and its envelope function F satisfies

$$E\left[\frac{Y_1^2(F)}{LL(Y_1(F))}\right] < \infty,$$

where

$$Y_1(f) = \sum_{N_1 < j \le N_2} f(X_j).$$

If

$$\sup_{f\in\mathcal{F}} Var\ Y_1(f) < \infty$$

then the bounded LIL hold over $\mathcal{F}$. If

$$\mathcal{F} \text{ is } \rho_\pi\text{-totally bounded,}$$

where the metric $\rho_\pi(f,g) = [E[\sum_{N_1<j\le N_2}(f-g)(X_j) - \pi(f-g)]^2]^{\frac{1}{2}}$, then the compact LIL holds over $\mathcal{F}$.

Let $\{X_n\}$ be a Markov chain wth ergodicity of degree 2 and let $\mathcal{F}$ be a countable family of functions on S satisfying

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty, \tag{5.1}$$

and assume its envelope function F satisfies

$$\int_C \pi(dx) E_x \max_{n \le \tau_C} \left(S_n^2(F)/LLS_n(F) \right) < \infty. \tag{5.2}$$

Here the notation is as in Chapter 4. If

$$\sup_{f \in \mathcal{F}} E[\sum_{N_1 < j \le N_2} f(X_j)]^2 < \infty \tag{5.3}$$

then from Theorem 5.10 below the bounded LIL holds over $\mathcal{F}$.

Chen [2, Theorem 4.3 of Chapter 3] has LIL results on ergodic Markov chains taking values in a separable Banach space B. Let $\{X_n\}$ be an ergodic Markov chain with a finite order. Then the necessary and sufficient conditions for the bounded LIL, i.e.

$$\limsup_{n \to \infty} \frac{\|S_n\|}{\sqrt{2nLLn}} < \infty \quad \text{a.s.},$$

are
(i) the CLT holds for every $f \in B^*$,
(ii) for some small set C,

$$\int_C \pi(dx) E_x \max_{n \le \tau_C} \left(\left\| S_n^2 \right\| / LL \left\| S_n \right\| \right) < \infty,$$

(iii) $\{S_n/\sqrt{2nLLn}\}$ is bounded in probability.

Let $\mathcal{F}$ be a family of bounded functions on S and define the metric

$$d(f,g) = \sup_{x \in S} |f(x) - g(x)|$$

If $\mathcal{F}$ is compact with respect to d, then $\mathcal{C}(\mathcal{F},\mathbf{R})$ with sup norm is a separable Banach space and we can consider the S valued Markov chain taking values in $\mathcal{C}(\mathcal{F},\mathbf{R})$ given by $X(f)(\cdot) = f(\cdot)$. Thus we are able to apply Chen's Theorem above. However, if $\mathcal{F}$ satisfies (5.1), then using Theorem 5.10 below we don't have to check Chen's conditon (iii) that $\{S_n/\sqrt{2nLLn}\}$ is bounded in probability.

Dudley [3] proved the above combinatorial condition (5.1) is satisfied in the case where the subgraphs of the functions in $\mathcal{F}$ are a VC class of set.

5.2 Uniform LIL for regenerative processes

Let $\{X_n\}$ be a regenerative process with renewal times N_i, taking values in S. We assume that $E(N_2 - N_1) < \infty$ and denote $\mu = E(N_2 - N_1)$ throughout the paper. Define

$$\pi(A) = \frac{1}{\mu} E\Big(\sum_{N_1 < j \le N_2} 1_A(X_j)\Big) \quad \text{for all } A \subseteq S.$$

Then π is a probability measure on S (called a steady state distribution). For all $f \in L^1(S, \pi)$

$$n^{-1} \sum_{k=1}^{n} f(X_k) \to \pi(f) \quad \text{a.s.}$$

Define the centered sum

$$S_n(f) = \sum_{j=1}^{n} \left(f(X_j) - \pi(f) \right),$$

and

$$Y_k(f) = \sum_{N_k < j \le N_{k+1}} f(X_j).$$

Then $\{Y_k(f)\}$ are i.i.d. and $EY_1(f) = \mu\pi(f)$. Define

$$LLx = \log\log(\max\{x, e^e\}).$$

Theorem 5.1. Suppose $E(N_2 - N_1)^2 < \infty$. Let $\mathcal{F}$ be a countable family of functions on S satisfying

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty, \tag{5.4}$$

$$\sup_{f \in \mathcal{F}} Var\ Y_1(f) < \infty, \tag{5.5}$$

and such that its envelope function F satisfies

$$E\left[\frac{Y_1^2(F)}{LL(Y_1(F))}\right] < \infty. \tag{5.6}$$

Then

$$\sup_{f\in\mathcal{F}}\frac{1}{\sqrt{2nLLn}}\sum_{k=1}^{n}\left[Y_k(f)-\mu\pi(f)\right]\to 0 \quad \text{in probability.}$$

Proof. We denote

$$S'=\bigcup_{n=1}^{\infty}S^n$$

and define

$$Z_k=(X_{N_k+1},...,X_{N_{k+1}})$$

for $k=1,2,3,..$ on S'. Let $P'=\mathcal{L}(Z_1)$, and P'_n be the n-th empirical measure of P'. For $f:S\to\mathbf{R}$, $f':S'\to\mathbf{R}$ is defined by

$$(y_1,...,y_n)\to\sum_{k=1}^{\infty}f(y_k).$$

Then $Y_k(f)=f'(Z_k)$. Define

$$\mathcal{F}'=\{f':f\in\mathcal{F}\},$$

and

$$v_n(f')=n^{-\frac{1}{2}}\sum_{k=1}^{n}\left[f'(Z_k)-Ef'(Z_1)\right].$$

Denote $b_n=\sqrt{2LLn}$ and $\|\cdot\|_{\mathcal{F}'}=\sup_{f\in\mathcal{F}}(\cdot)(f')$. We need to show that

$$\left\|\frac{v_n}{b_n}\right\|_{\mathcal{F}'}\to 0 \quad \text{in probability.}$$

Let $\{\varepsilon_i\}$ be a Rademacher sequence, define

$$v_n^0(f')=n^{-\frac{1}{2}}\sum_{k=1}^{n}\varepsilon_k f'(Z_k).$$

By Lemma 5.1 in Alexander and Talagrand [1] we have the following lemma.

Lemma 5.2. For all $\eta>0$ and $\alpha^2\geq\sup_{f'\in\mathcal{F}'}Var(f'(Z_1))$, we have

$$P\left[\|v_n\|_{\mathcal{F}'}>\eta\right]\leq 2P\left[\left\|v_n^0\right\|_{\mathcal{F}'}>\eta/2-\alpha\right].$$

So we can replace v_n by v_n^0.

Let P_X, E_X and $P_\varepsilon, E_\varepsilon$ denote the probability and expection with respect to the sequence $\{X_i\}$ and $\{\varepsilon_i\}$. The probability space on $\{X_i\}$ and $\{\varepsilon_i\}$ are defined as a product space $(\Omega_1, \mathcal{A}_1, P_X) \times (\Omega_2, \mathcal{A}_2, P_\varepsilon)$. Let d be a pseudometric on $\mathcal{F}$ and $N(\varepsilon, \mathcal{F}, d)$ be the minimum m for which there exists $g_1, ..., g_m \in \mathcal{F}$ such that for all $f \in \mathcal{F}$, $d(f, g_i) < \varepsilon$ for some $1 \leq i \leq m$. By Lemma 5.2 in Alexander and Talagrand [1] we have the following lemma.

Lemma 5.3. There exists a universal contant R_1 such that for all $f_0 \in \mathcal{F}'$

$$E_\varepsilon \left\| v_n^0 \right\|_{\mathcal{F}'} \leq R_1 \left[\int_0^{\Delta_n(\mathcal{F}')} \left(\ln N(u, \mathcal{F}', e_{P_n'}) \right)^{\frac{1}{2}} du + \Delta_n(\mathcal{F}') + \left(P_n'(f_0^2) \right)^{\frac{1}{2}} \right]$$

where $e_{P_n'}(f', g') = [n^{-1} \sum_{k=1}^n (f' - g')^2 (Z_k)]^{\frac{1}{2}}$ and $\Delta_n(\mathcal{F}')$ is the diameter of $\mathcal{F}'$ for metric $e_{P_n'}$.

The following lemma is from the proof of Lemma 4.2 in Levental [6].(Also see the proof of Lemma 4.4 in chapter 4, there $N_2(u, \mathcal{F}', P_n') = N(u, \mathcal{F}', e_{P_n'})$.)

Lemma 5.4. Suppose $E(N_2 - N_1)^2 < \infty$. Then for $u > 0$ and $\lambda > 0$ we have

$$P(N(u, \mathcal{F}', e_{P_n'}) > N(u/2[E(N_2 - N_1)^2]^{\frac{1}{2}}, \mathcal{F})) < \lambda$$

for n large enough.

¿From Lemma 5.4 in Alexander and Talagrand [1].

Lemma 5.5. If

$$E \left[\frac{Y_1^2(F)}{LL(Y_1(F))} \right] < \infty$$

then

$$\frac{1}{LLn} P_n'(F'^2) \to 0 \quad \text{in probability.}$$

Proof of Theorem 5.1. Let $\delta, \theta > 0$ and let

$$G_n = \{\omega_1 \in \Omega_1 : [P_n'(F'^2)]^{\frac{1}{2}} \leq \theta b_n\}.$$

Then by Lemma 5.2, for large n

$$\begin{aligned} P\left[\left\|\frac{v_n}{b_n}\right\|_{\mathcal{F}'} > \delta\right] &\leq 2P\left[\left\|\frac{v_n^0}{b_n}\right\|_{\mathcal{F}'} > \frac{\delta}{2}\right] \\ &\leq 2P_X(G_n^c) + 2\sup_{\omega_1\in\Omega_1} P_\varepsilon\left[\left\|\frac{v_n^0}{b_n}\right\|_{\mathcal{F}'} > \frac{\delta}{2}\right] \\ &\leq 2P_X(G_n^c) + \frac{4}{\delta b_n}\sup_{\omega_1\in\Omega_1} E_\varepsilon \|v_n^0\|_{\mathcal{F}'} \end{aligned} \tag{5.7}$$

Clearly, $\Delta_n(\mathcal{F}') \leq [P_n'(F'^2)]^{\frac{1}{2}}$, and from Lemma 5.4 outside a set of small probability λ we have

$$\int_0^{\Delta_n(\mathcal{F}')} \left(\ln N(u, \mathcal{F}', e_{P_n'})\right)^{\frac{1}{2}} du \leq \int_0^\infty \left[\ln N\left(\frac{u}{2[E(N_2-N_1)^2]^{\frac{1}{2}}}, \mathcal{F}\right)\right]^{\frac{1}{2}} du = R_2 < \infty.$$

Thus for $\omega_1 \in G_n$, from Lemma 5.3

$$E_\varepsilon \left\|v_n^0\right\|_{\mathcal{F}'} \leq R_1[R_2 + 2\theta b_n + \theta b_n],$$

and hence by (5.7), for large n

$$P\left[\left\|\frac{v_n}{b_n}\right\|_{\mathcal{F}'} > \delta\right] \leq 2P_X(G_n^c) + \frac{4R_1R_2}{\delta b_n} + \frac{12\theta R_1}{\delta} + \lambda.$$

Since θ, λ are arbitrary and $P_X(G_n^c) \to 0$ by Lemma 5.5, we have

$$P\left[\left\|\frac{v_n}{b_n}\right\|_{\mathcal{F}'} > \delta\right] \to 0 \quad \text{as } n \to \infty \text{ in probability.}$$

Theorem 5.6. Suppose $E(N_2 - N_1)^2 < \infty$. Let $\mathcal{F}$ be a countable family of functions on S satisfying

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty$$

and such that its envelope function F satisfies

$$E\left[\frac{Y_1^2(F)}{LL(Y_1(F))}\right] < \infty.$$

If

$$\sup_{f\in\mathcal{F}} Var\ Y_1(f) < \infty,$$

then the bounded LIL holds over $\mathcal{F}$, i.e.

$$\sup_n \sup_{f\in\mathcal{F}} \frac{\mid S_n(f) \mid}{\sqrt{2nLLn}} < \infty \quad \text{a.s..}$$

If

$$\mathcal{F} \text{ is } \rho_\pi\text{-totally bounded,} \tag{5.8}$$

where the metric $\rho_\pi(f,g) = [E[\sum_{N_1<j\le N_2}(f-g)(X_j) - \pi(f-g)]^2]^{\frac{1}{2}}$. Then the compact LIL holds over $\mathcal{F}$.

Remark. Arcones obtained an empirical process compact LIL over VC graph classes $\mathcal{F}$ under the condition that the envelope function of $\mathcal{F}$ is in L^p for some $2 < p < \infty$ and the β-mixing coefficients satisfy

$$\sum_{k=1}^{\infty} k^{2/(p-2)}(\log k)^{2(p-1)/(p-2)}(\log\log n)^{-(2p-1)/(p-2)}\beta_k < \infty.$$

But in Example 4.10 of chapter 4 the conditions of Theorem 5.6 hold and the envelope function of $\mathcal{F}$ is not in L^p for any $p > 2$, and Arcones' conditions fail.

To prove Theorem 5.6 we define

$$R_n(f) = \sum_{1\le j\le N_1 \text{ or } N_{i(n)}<j\le n} (f(X_j) - \pi(f))$$

where $l(n) = \max\{k : N_k \le n\}$. Then

$$S_n(f) = \sum_{k=1}^{l(n)-1} (Y_k(f) - (N_{k+1} - N_k)\pi(f)) + R_n(f).$$

Lemma 5.7. If $E(N_2 - N_1)^2 < \infty$ and

$$E\left[\frac{Y_1^2(F)}{LL(Y_1(F))}\right] < \infty$$

then

$$\sup_{f\in\mathcal{F}} \frac{\mid R_n(f) \mid}{\sqrt{2nLLn}} \to 0 \quad \text{a.s..}$$

proof. We have

$$\sup_{f\in\mathcal{F}}\frac{|\sum_{1\le j\le N_1}(f(X_j)-\pi(f))|}{\sqrt{2nLLn}}\le\frac{\sum_{1\le j\le N_1}F(X_j)+N_1\pi(F)}{\sqrt{2nLLn}}\to 0\quad\text{a.s..}$$

and

$$\frac{(n-N_{l_{(n)}})\pi(F)}{\sqrt{2nLLn}}\to 0\quad\text{a.s..}$$

from $E(N_2-N_1)^2<\infty$. Thus we only have to show that

$$\frac{\sum_{N_{i(n)}<j\le n}F(X_j)}{\sqrt{2nLLn}}\to 0\quad\text{a.s..}$$

It's enough to show

$$\frac{Y_n(F)}{\sqrt{2nLLn}}\to 0\quad\text{a.s..}$$

We have

$$E\left[\frac{Y_1^2(F)}{\varepsilon LL(Y_1(F))}\right]<\infty$$

for all $\varepsilon>0$. Thus

$$\sum_{n=1}^{\infty}P\left(\frac{Y_n^2(F)}{LL(Y_n(F))}\ge n\varepsilon\right)=\sum_{n=1}^{\infty}P\left(\frac{Y_1^2(F)}{\varepsilon LL(Y_1(F))}\ge n\right)<\infty.$$

By Borel-Cantelli lemma, we have

$$P\left(\frac{Y_n^2(F)}{LL(Y_n(F))}\ge n\varepsilon\text{ i.o.}\right)=0,$$

that is

$$P\left(\frac{Y_n^2(F)}{LL(Y_n(F))}<n\varepsilon\text{ eventually}\right)=1.$$

Thus

$$P\left(\frac{Y_n^2(F)}{LLn}<n\varepsilon\text{ eventually}\right)=1.$$

So

$$\overline{\lim_n}\frac{Y_n^2(F)}{LLn}\le\varepsilon\text{ a.s.}$$

for all $\varepsilon>0$.

Proof of Theorem 5.6. Applying Theorem 9 in Ledoux and Talagrad [4] and [5] to $\{Z_k\}$, we have that the bounded LIL hold over $\mathcal{F}'$ with respect to $\{Z_k\}$ by Theorem 5.1. i.e.

$$\sup_n \sup_{f\in\mathcal{F}} \frac{1}{\sqrt{2nLLn}} \left| \sum_{k=1}^{n} (f'(Z_k) - \mu\pi(f)) \right| < \infty \quad \text{a.s..}$$

This is equivalent to

$$\sup_n \sup_{f\in\mathcal{F}} \frac{1}{\sqrt{2nLLn}} \left| \sum_{k=1}^{i(n)-1} (f'(Z_k) - (N_{k+1} - N_k)\pi(f)) \right| < \infty \quad \text{a.s..} \tag{5.9}$$

since $n/l(n) \to \mu$ a.s. and

$$\sup_n \sup_{f\in\mathcal{F}} \frac{|\pi(f)|}{\sqrt{2nLLn}} \left| \sum_{k=1}^{n} [(N_{k+1} - N_k) - \mu] \right| < \infty.$$

¿From (5.9) and Lemma 5.7 we have the bounded LIL hold over $\mathcal{F}$. The proof of the compact LIL is similar.

5.3 Uniform LIL for Markov chains

Definition 5.8. Using the notations of section 2, we call $\{X_i\}$ a **weakly regenerative process** if $N_i < \infty$ almost surely for every $i \geq 1$ and if for every $f : \Omega \to \mathbf{R}$ which is bounded and $\Sigma/\mathcal{B}(\mathbf{R})$ measurable,

$$E[f(\theta_{N_{i+1}}) \mid \mathcal{G}_{N_i} \vee \sigma(N_{i+1} - N_i)] = E[f(\theta_{N_1})].$$

The following two properties of the process $\{W_i\}$ are equivalent to the above definition:
(i) The post $N_{i+1} + 1$ process is independent of the occurence up to and including N_i, and
(ii) $\mathcal{L}((W_{N_1+1}, ...)) = \mathcal{L}((W_{N_i+1}, ...)) \quad$ for all $i = 1, 2, ...$

We put $S^* = S \cup \{*\}$ where $*$ denotes an ideal point which is not in S. We define two processes $\{E_i\}$ and $\{O_i\}$, taking values in S^*:

$O_i = X_i$ if $N_{2k+1} < i \le N_{2k+2}$ for some $k \ge 0$ and $O_i = *$ otherwise
$E_i = X_i$ if $N_{2k} < i \le N_{2k+1}$ for some $k \ge 0$ and $E_i = *$ otherwise.
Then $\{O_i\}$ and $\{E_i\}$ are regenerative processes taking values in S^*, with renewal times $\{N_{2i}\}_{i\ge1}$ and $\{N_{1i+1}\}_{i\ge1}$ respectively. Every function $f : S \to \mathbf{R}$ will be considered as defined on S^* with the identification $f(*) = 0$.

Lemma 5.9. Let $\{X_i\}$ be a weakly regenerative process with renewal times N_i satisfying $E(N_2 - N_1)^2 < \infty$. Let $\mathcal{F}$ be a countable family of functions on S with envelope function F. If

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty,$$

$$\sup_{f\in\mathcal{F}} E[\sum_{N_1<j\le N_2} f(X_j)]^2 < \infty, \tag{5.10}$$

and its envelope function F satisfies

$$E\left[\frac{[\sum_{N_1<j\le N_2} F(X_j)]^2}{LL\sum_{N_1<j\le N_2} F(X_j)}\right] < \infty, \tag{5.11}$$

then the bounded LIL holds over $\mathcal{F}$.
Proof. Since $f(X_i) = f(O_i) + f(E_i)$, the proof follows from Theorem 5.6.

Using the split chain technique [2, chapter 1], ergodic Markov chains can be embedded in a weakly regenerative structure. That is, if $\{X_i\}$ is a ergodic Markov chain, then by Theorem 2.2 of chapter 1 in Chen, $\{X_i\}$ is a weakly regenerative process with renewal times

$$N_i = m\tau(i) + m - 1,$$

where $\tau(i)$ is defined in [2, (2.19) of chapter 1] and m is the order of some fixed small set.

¿From Proposition 5.16 of Nummelin and Theorem 4.6 of Chapter 1 in Chen (with $\varphi(s) = s^2$ and $\xi \equiv 1$), ergodicity of degree 2 implies $E(N_2 - N_1)^2 < \infty$. Applying Lemma 5.9 to Markov chains, we have the following theorem.

Theorem 5.10. Let $\{X_i\}$ be a Markov chain with ergodic of degree 2. Let

$\mathcal{F}$ be a countable family of functions on S with envelope function F such that

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty,$$

$$\sup_{f \in \mathcal{F}} E[\sum_{N_1 < j \leq N_2} f(X_j)]^2 < \infty,$$

and its envelope function F satisfies

$$\int_C \pi(dx) E_x \max_{n \leq \tau_C} \left(S_n^2(F)/LLS_n(F) \right) < \infty, \tag{5.12}$$

Then the bounded LIL holds over $\mathcal{F}$.

Proof. We have to show (5.11). Using Theorem 4.6 of chapter 1 in Chen with $\xi \equiv F$ and $\varphi(s) = s^2/LLs$, (5.12) implies

$$E\left[\frac{[\sum_{N_1 < j \leq N_2} (F(X_j) - \pi(F))]^2}{LL \left| \sum_{N_1 < j \leq N_2} (F(X_j) - \pi(F)) \right|} \right] < \infty.$$

Let $Y = \sum_{N_1 < j \leq N_2} F(X_j)$ and $Y' = \sum_{N_1 < j \leq N_2} (F(X_j) - \pi(F)) = Y - (N_2 - N_1)\pi(F)$. Then

$$\begin{array}{rcl} E\left(\frac{Y^2}{LLY}\right) & = & E\left(\frac{Y^2}{LLY} 1_{\{Y' \geq 0\}}\right) + E\left(\frac{Y^2}{LLY} 1_{\{Y' < 0\}}\right) \\ & \leq & E\left(\frac{2Y'^2 + 2(N_2 - N_1)^2 \pi^2(F)}{LL|Y'|} 1_{\{Y' \geq 0\}}\right) + E\left((N_2 - N_1)^2 \pi^2(F) 1_{\{Y' < 0\}}\right) \\ & < & 0. \end{array}$$

In Theorem 2.2 of chapter 3 of Chen if $f \in L^2(\pi)$ and

$$\limsup_n \frac{|S_n(f)|}{\sqrt{2nLLn}} < \infty \text{ a.s.}$$

then

$$n^{-\frac{1}{2}} S_n(f) \Longrightarrow N(0, \sigma_f^2).$$

In the case the order of the Markov chain $m = 1$ we have from Theorem 2.2 of chapter 2 in Chen

$$\sigma_f^2 = c \cdot E[\sum_{N_1 < j \leq N_2} f(X_j)]^2.$$

Thus we have the following Corollary.

Corollary 5.11. Let $\{X_i\}$ be a Markov chain with ergodic of degree 2 and order 1. Let $\mathcal{F}$ be a countable family of functions on S with envelope function F such that

$$\int_0^\infty [\log N_2(\varepsilon, \mathcal{F})]^{\frac{1}{2}} d\varepsilon < \infty,$$

$$\sup_{f \in \mathcal{F}} \sigma_f^2 < \infty,$$

and its envelope function F satisfies

$$\int_C \pi(dx) E_x \max_{n \leq \tau_C} \left(S_n^2(F)/LLS_n(F) \right) < \infty,$$

Then the bounded LIL holds over $\mathcal{F}$.

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