PRICING ASIAN OPTIONS

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1. Introduction

We see randomness everywhere in nature. Statisticians use mathematical tools to make sense of this randomness, often representing it with something called a "probability distribution. " A probability distribution is simply a way to describe the likelihood of different outcomes. For a set of data, it shows us how the data points are spread out and which values are more or less common.

Finding the right distribution usually involves two steps. First, we visualize the data. We might use a histogram, which is like a bar chart showing how many data points fall into different ranges, or a density plot, which is a smooth curve showing the same information. Second, we compare the shape of our data's visualization to some common, well-known distributions to see which one fits best. For example, if the data are evenly spread around a central point and look like a bell curve, it might follow a "normal distribution."

Sometimes, the best-fitting distribution is too complicated to easily use for calculations. To get around this, statisticians use a technique called "moment matching." This technique simplifies a complex distribution by approximating it with a simpler one (like the Gaussian, Gamma, or Pearson distribution). The key is to make sure that certain important characteristics, called "moments," of the simpler distribution match those of the original, more complex one. These moments include mean (average), variance (spread), skewness (asymmetry) and kurtosis (tailed heaviness / peak). By matching these moments, we can use a simpler distribution that's easier to work with, while still capturing the important features of the original data.

This article presents a simple moment-matching technique to ascertain the probability distribution of a specific quantity and to assess the valuation of an "Asian option," which will be detailed in the following sections. However, we will first provide a brief overview of the stock market and discuss various mathematical models related to stock price dynamics.

2. Market Models

A stock price is the cost of buying one share of a company. It represents how much people think that share is worth based on the company's performance, future potential, and overall market conditions. Stock prices don't stay the same—they go up and down throughout the day based on factors like: (i) Company news (e.g., good earnings reports, new products, or scandals), (ii) market trends (e.g., the economy growing or slowing down), (iii) supply and demand (e.g., if many people want to buy the stock, its price rises. For example, the stock price dynamics of Taiwan Semiconductor Manufacturing Company Limited (TSMC) from 14th February 2024 to 10th February 2025 is depicted in the figure 1.

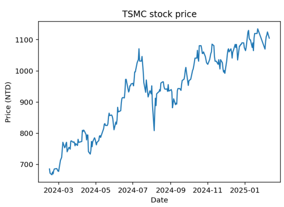

Figure 1. TSMC stock price

Stock prices are unpredictable and influenced by many factors. Modeling stock prices is like creating a map to help understand or predict their behavior. While no model can perfectly predict the future, they can give us useful insights, like: (i) Estimating risk: How much might we lose or gain? (ii) Making decisions: When is a good time to buy or sell? (iii) Simulating outcomes: What might happen under different scenarios?

Stock price modeling is based on the idea that prices move randomly, but with some patterns. Here are a few ways stock prices are modeled:

(i) **Random Walk Theory**: Stock prices move randomly, as someone flips a coin to decide whether to take a step forward or backward.
(ii) **Geometric Brownian Motion (GBM)**: This is a more advanced version of the random walk where stock prices grow over time but fluctuate randomly around that growth. GBM is used in financial models like the Black-Scholes model to price options.
(iii) **Mean Reversion Models**: If a stock price moves too high or too low, it eventually "reverts" to an average value (its "fair price").

The following figure 2 illustrates the trajectories of a random walk, geometric Brownian motion, and a mean reversion process.

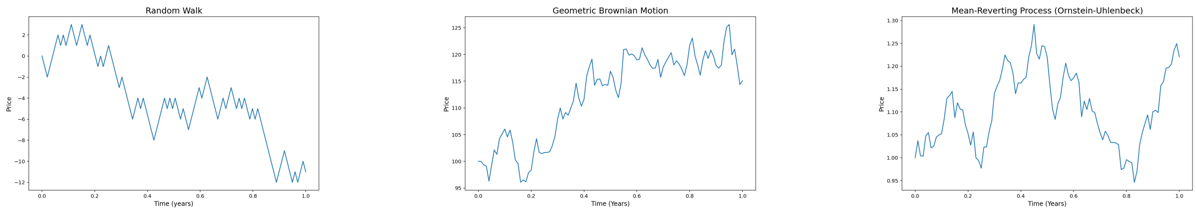

FIGURE 2. Random walk, GBM, OU processes

Next, we will examine the fundamental concepts of continuous-time Markov chain (CTMC) approximation, as this constitutes the central theme of this report.

3. Continuous time Markov Chain Approximation

Imagine a stock price. It jumps up and down throughout the day, seemingly at random. We might want to model this behavior to predict future prices or understand the risks involved. One way to do this is with a Markov chain. A Markov chain is a mathematical model that describes a sequence of possible events where the probability of each event depends only on the state attained in the previous event. One can think of it like a game where what happens next depends only on where we are right now, not on how we got there.

Now, a continuous-time Markov chain (CTMC) is a special kind of Markov Chain where these events can happen at any moment, not just at specific intervals. Our stock price example fits this pretty well – the price can change at any second, not just at the beginning or end of each trading day.

Assume we postulate that the market exhibits dynamics as discussed in section 2, which may be inherently complex. Fortunately, as noted in Mijatović and Pistorius (2013), if we possess a complex model (S) that characterizes asset prices, it is possible to derive a simplified representation through a Continuous-Time Markov Chain (CTMC) approximation (X) that exhibits analogous behavior. To achieve this, we substitute the intricate dynamics of S with a Markov chain, where the transition probabilities are informed by an approximation of the governing rules of S. Consequently, the resulting model X retains the essential characteristics, or probabilistic structure, of S, while being more tractable for analysis.

Here's how it works in the stock market context:

(i) **States**: We define different "states" for the stock price. For example, we could have states like "price below 100," "price between 100 and 110," "price between 110 and 120," and so on. These states represent ranges of possible stock prices.

(ii) **Transitions**: We assume the stock price can jump between these states. For example, it might jump from the "$100 - 110$" state to the "$110 - 120$" state (price goes up) or to the "below 100" state (price goes down). This can done using the dynamics of S.
(iii) **Transition Rates**: Crucially, we assign rates to these transitions. A transition rate tells us how often we expect the stock price to jump from one state to another. For example, we might estimate that the transition rate from "$100 - 110$" to "$110 - 120$" is higher than the rate from "$100 - 110$" to "below 100," reflecting an upward trend. This is also achived via the dynamics of S.
(iv) **Approximation**: We use these states and transition rates to build our CTMC. This CTMC approximates the real stock price movements. It's not perfect, but it can give us a simplified, manageable model.

Why do we approximate? The reason is that the real stock market is too complex to model exactly. Using a CTMC approximation, we can:

i. **Make predictions**: We can use the model to estimate the probability of the stock price being in a certain range at some point in the future.
ii. **Assess risk**: We can use the model to understand the likelihood of large price swings.
iii. **Simplify calculations**: The CTMC, while a simplification, is still a mathematical model that we can analyze. It's much easier to work with than trying to model every single factor that influences stock prices.

It's important to remember that this is an approximation. The accuracy of the model depends on how well our chosen model reflects the real stock market behavior. A good model requires careful analysis and often involves statistical techniques to estimate the transition rates from historical data. Still, the CTMC provides a powerful and widely-used tool for understanding and working with the randomness of stock prices.

The implementation of the Continuous-Time Markov Chain (CTMC) algorithm is detailed in the following section.

4. Asian Option price

Imagine we want to buy a stock at some point in the future, but we're worried about the price fluctuating too much. An Asian option can help!

Instead of locking in a price today, an Asian option lets us buy the stock at the average price over a certain period. This smooths out the price swings, so we're not as exposed to sudden price drops right before we buy. We can think of it like this:

i. **European option**: we agree to buy a stock for 100 in a month. If the price is 110 then, we're happy! But if it's 90, we're stuck buying at 100.
ii. **Asian option**: we agree to buy the stock for the average price over the next month. If the price jumps around a lot, the average will be somewhere in the middle, reducing our risk.

Essentially, an Asian option gives us a chance to buy at a more stable, average price, making it a less risky way to invest in a stock.

4.1. **Mathematics behind pricing.** Option pricing is all about figuring out the fair value of an option, which is a financial contract that gives you the right (but not the obligation) to buy or sell something (like a stock) at a specific price before a certain date.

To see the mathematical formulation of option pricing, we heighlight the key Factors in Option Pricing:

(i) Current stock price (S): How much the stock is worth right now.
(ii) Strike price (K): The price at which you can buy (call option) or sell (put option) the stock.
(iii Time to expiration (T): How much time is left before the option expires.

(iv Volatility (σ): How much the stock price moves up and down (more uncertainty = higher option price).

(v) Risk-free rate (r): The interest rate for a "risk-free" investment, like government bonds.

The value of an option is based on what we could expect to gain from it in the future, adjusted for the fact that money today is worth more than money tomorrow (time value of money). Mathematically, we write the European call option price $C(S)$ as follows:

$$C(S) = E\left[e^{-rT}\max\{S(T) - K, 0\}\right],$$

where E denotes the mean value and $S(T)$ denotes the stock price at a future time T. If one knows the probability distribution of $S(T)$, it is easy for them to compute $C(S)$. As mentioned earlier, for Asian option depends on the average of the complete path the expression for the call option price becomes

$$C(S) = E\left[e^{-rT}\max\{\frac{1}{T}\int_0^T S(t)dt - K, 0\}\right].$$

K	Benchmark	Moment-based method	Error
			Lognormal
2	0.05598604	0.055805831	-1.8E-04
2	0.21838755	0.218290695	-9.7E-05
2	0.19317379	0.19366318	4.9E-04
2	0.24641569	0.246850384	4.3E-04
2	0.30622036	0.30654161	3.2E-04

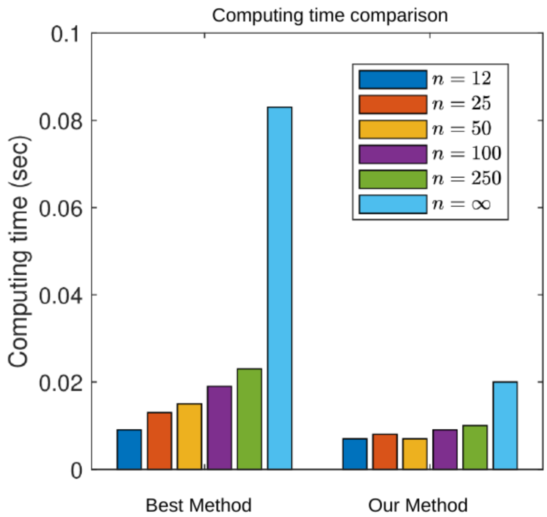

FIGURE 3. The left table presents the accuracy of our methodology applied to the Geometric Brownian Motion model, utilizing parameter sets (r, σ, S_0) that belong to the set $\{(0.02, 0.1, 2), (0.18, 0.3, 2), (0.05, 0.5, 1.9), (0.05, 0.5, 2), (0.05, 0.5, 2.1)\}$. The right figure illustrates the computational efficiency of our method, demonstrating its superiority over our nearest competitor, labelled as 'Best method'.

In this scenario, it suffices to compute the probability distribution of the quantity $A = \frac{1}{T}\int_0^T S(t)dt$. As the reader may have observed, the distribution of A poses significant challenges for computation. Nevertheless, previous work utilizing the Continuous-Time Markov Chain (CTMC) algorithm for general Markov chains has produced the moment generating function (MGF) of A as documented in Cai et al. (2015). The Moment Generating Function is a fundamental tool in probability and statistics, employed to analyze random variables. It serves as a comprehensive summary, allowing for the generation of all moments of a random variable through a single, elegant formula, which is the rationale behind its designation. Notably, if the MGF is known, the moments can be extracted by differentiating the MGF with respect to t and evaluating at $t = 0$.

Utilizing the MGF derived in Cai et al. (2015), we have proposed a method to efficiently compute the requisite moments—namely, the mean, variance, skewness, and kurtosis—in our work presented in Das et al. (2022). With these moments determined, we apply the method of moment matching to approximate the probability distribution of the average process A. It is noteworthy that the Pearson approximation for the probability distribution demonstrates excellent performance. Our methodology is also computationally more efficient than existing alternatives. The subsequent table and figure 3 illustrate the accuracy and computational time associated with our approach.

Conclusion

In this article, we have explored fundamental concepts of probability, simulation—specifically the Continuous-Time Markov Chain (CTMC) algorithm—and option pricing. While these concepts may not be presented with mathematical rigor, we aim to provide a more accessible perspective. As illustrated in Figure 3, our method delivers a favorable balance between speed and accuracy, which can be applied to other significant areas of interest within financial engineering.

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