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Rate of Poisson approximation for nearest neighbor counts in large-dimensional Poisson point processes

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ABSTRACT

Consider two independent homogeneous Poisson point processes Π of intensity λ and Π' of intensity λ' in d-dimensional Euclidean space. Let $q_{k,d}$, $k = 0, 1, \ldots$, be the fraction of Π-points which are the nearest Π-neighbor of precisely k Π'-points. It is known that as $d \to \infty$, the $q_{k,d}$ converge to the Poisson probabilities $e^{-\lambda'/\lambda}(\lambda'/\lambda)^k/k!$, $k = 0, 1, \ldots$. We derive the (sharp) rate of convergence $d^{-1/2}(4/3\sqrt{3})^d$, which is related to the asymptotic behavior of the variance of the volume of the typical cell of the Poisson–Voronoi tessellation generated by Π. An extension to the case involving more than two independent Poisson point processes is also considered.

In this note, we consider two independent homogeneous Poisson point processes Π of intensity λ and Π' of intensity λ' in d-dimensional Euclidean space $\mathbb{R}^d$. Let $p_{k,d}$, $k = 0, 1, \ldots,$ be the fraction of Π-points which are the nearest Π-neighbor of precisely k other Π-points, and $q_{k,d}$, $k = 0, 1, \ldots,$ the fraction of Π-points which are the nearest Π-neighbor of precisely k Π'-points. Here for given $\mathbf{v} \in \mathbb{R}^d$, a point $Q \in \Pi$ is called the nearest Π-neighbor of $\mathbf{v}$ if $\|Q - \mathbf{v}\|_d < \|\mathbf{u} - \mathbf{v}\|_d$ for all $\mathbf{u} \in \Pi \setminus \{\mathbf{Q}\}$ where $\|\cdot\|_d$ denotes the Euclidean norm in $\mathbb{R}^d$. By ergodic-type arguments, $p_{k,d}$ and $q_{k,d}$ are well defined. In their Theorems 5 and 10, Newman et al. (1983) proved that

$$\lim_{d\to\infty} p_{k,d} = e^{-1}/k! \quad \text{and} \quad \lim_{d\to\infty} q_{k,d} = e^{-\rho}\rho^k/k!, \tag{1}$$

where $\rho = \lambda'/\lambda$. (See also Newman and Rinott, 1985.)

The limit results in (1) can also be formulated in terms of a "typical point" Q of Π, to be translated so that $Q = \mathbf{0} = (0, \ldots, 0)$, the origin in $\mathbb{R}^d$. We will refer to $\mathbf{0}$ as the typical point of Π. (Note that since Π is a Poisson point process, its Palm distribution at $\mathbf{0}$ is equivalent to the distribution of Π with an independently added point at $\mathbf{0}$; see e.g. Daley and Vere-Jones (2007, Proposition 13.1.VII).) Let M_d be the number of Π-points which have $\mathbf{0}$ as their nearest Π-neighbor, and $N_{\rho,d}$ the number of Π'-points which have $\mathbf{0}$ as their nearest Π-neighbor, i.e.

$$\begin{aligned} M_d &= \#\{\mathbf{u} \in \Pi : \|\mathbf{u} - \mathbf{0}\|_d < \|\mathbf{u} - \mathbf{v}\|_d \text{ for all } \mathbf{v} \in \Pi \setminus \{\mathbf{u}\}\}, \\ N_{\rho,d} &= \#\{\mathbf{u}' \in \Pi' : \|\mathbf{u}' - \mathbf{0}\|_d < \|\mathbf{u}' - \mathbf{v}\|_d \text{ for all } \mathbf{v} \in \Pi\} \\ &= \#(\Pi' \cap C_d), \end{aligned} \tag{2}$$

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where

$$C_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{0}\|_d < \|\mathbf{x} - \mathbf{u}\|_d \text{ for all } \mathbf{u} \in \Pi\}, \tag{3}$$

the (typical) Voronoi cell centered at $\mathbf{0}$ generated by $\Pi \cup \{\mathbf{0}\}$ (cf. Okabe et al., 2000). Note that $\mathcal{L}(M_d)$, the distribution of M_d, is independent of λ and λ', and $\mathcal{L}(N_{\rho,d})$ depends on λ and λ' only through $\rho = \lambda'/\lambda$. Then (1) is equivalent to the limit results that as $d \to \infty$, M_d and $N_{\rho,d}$ converge in distribution to $Po(1)$ and $Po(\rho)$, respectively, where $Po(\rho)$ denotes the Poisson distribution with mean ρ.

In the present note, we derive the rate of convergence for $N_{\rho,d}$ as stated below.

Theorem 1. *For any given $\rho_0 > 0$, there exists a constant $c_1(\rho_0) > 0$ such that for all $1 \le d < \infty$ and $0 < \rho \le \rho_0$,*

$$c_1(\rho_0)\rho^2 \frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \le d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \le c_2\rho(1-e^{-\rho})\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d, \tag{4}$$

where d_{TV} denotes the total variation distance and $c_2 > 0$ is a constant (independent of ρ_0).

Proof. Without loss of generality, assume $\lambda = 1$, so that $\rho = \lambda'/\lambda = \lambda'$. Note that $N_{\rho,d}$ has a mixed Poisson distribution. By (2) and (3), the conditional distribution of $N_{\rho,d}$ given $\mu_d(C_d) = v$ is $Po(\lambda' v) = Po(\rho v)$ where $\mu_d(S)$ denotes the d-dimensional Lebesgue measure (volume) of measurable $S \subset \mathbb{R}^d$. Also, $\mathrm{E}[\mu_d(C_d)] = 1/\lambda = 1$, and by Alishahi and Sharifitabar (2008, Theorem 3.1),

$$\mathrm{var}(\mu_d(C_d)) \le \frac{c_2}{\lambda^2}\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d = c_2\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } 1 \le d < \infty,$$

for some constant $c_2 > 0$. By Barbour et al. (1992, Theorem 1.C(ii)), we have

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\le \rho^{-1}(1-e^{-\rho})\mathrm{var}(\rho\mu_d(C_d)) \\ &\le c_2\rho(1-e^{-\rho})\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

establishing the upper bound.

To derive the lower bound, fix a (large) $u > 0$ and consider the ball of volume u centered at $\mathbf{0}$, denoted by $B = B_{u,d}$. Applying Lemma 1 below with $\alpha_1 = \rho\mu_d(C_d \cap B) \le \rho u$, $\alpha_2 = \rho\mu_d(C_d \setminus B)$, $\beta_1 = \mathrm{E}[\alpha_1] = \rho\mathrm{E}[\mu_d(C_d \cap B)] \le \rho u$, $\beta_2 = \mathrm{E}[\alpha_2] = \rho\mathrm{E}[\mu_d(C_d \setminus B)] = \rho - \beta_1$, we have

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\ge P(N_{\rho,d} = 0) - e^{-\rho} \\ &= \mathrm{E}[e^{-\rho\mu_d(C_d)}] - e^{-\rho} \\ &= \mathrm{E}\{e^{-\alpha_1-\alpha_2} - e^{-\beta_1-\beta_2}\} \\ &\ge \mathrm{E}\{e^{-\beta_1-\beta_2}[h_1(\rho u)(\alpha_1-\beta_1)^2 - (\alpha_1-\beta_1)] - e^{-\alpha_1-\beta_2}(\alpha_2-\beta_2)\} \\ &= e^{-\rho}h_1(\rho u)\mathrm{var}(\alpha_1) - e^{-\beta_2}\mathrm{E}[e^{-\alpha_1}(\alpha_2-\beta_2)] \\ &\ge e^{-\rho}h_1(\rho u)\mathrm{var}(\rho\mu_d(C_d \cap B)) \\ &\ge e^{-\rho}h_1(\rho u)\rho^2 c\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

where the last two inequalities follow from Lemma 2 (with $S_1 = B$ and $S_2 = \mathbb{R}^d \setminus B$) and Lemma 3 (with $u \ge u_0$ and $c > 0$ and $u_0 < \infty$ appearing in the statement of Lemma 3). Noting that h_1 is nonincreasing, we have for all $1 \le d < \infty$ and $0 < \rho \le \rho_0$,

$$\begin{aligned} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) &\ge ce^{-\rho_0}h_1(\rho_0 u)\rho^2\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \\ &= c_1(\rho_0)\rho^2\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d, \end{aligned}$$

where $c_1(\rho_0) = ce^{-\rho_0}h_1(\rho_0 u)$. The proof is complete. □

Remark 1. The lower and upper bounds in (4) may be expressed as

$$\inf_{1\le d<\infty, 0<\rho\le\rho_0} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \Bigg/ \left[\rho^2\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d\right] > 0,$$

$$\sup_{1\le d<\infty, \rho>0} d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \Bigg/ \left[\rho(1-e^{-\rho})\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d\right] < \infty.$$

Since $N_{\rho,d}$ has a mixed Poisson distribution, we may apply Barbour et al. (1992, Theorem 3.F) to obtain a lower bound for $d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho))$, which, however, involves the third and fourth moments of $\mu_d(C_d)$. As no good estimates of these higher-order moments are available, we make use of Lemmas 1–3 to derive the lower bound in (4) which only involves the second moment of $\mu_d(C_d \cap B_{u,d})$.

Lemma 1. *For $u > 0$, $0 \le \alpha_1, \beta_1 \le u$, and $\alpha_2, \beta_2 \in \mathbb{R}$, we have*

$$e^{-\alpha_1-\alpha_2} - e^{-\beta_1-\beta_2} \ge e^{-\beta_1-\beta_2}[h_1(u)(\alpha_1-\beta_1)^2 - (\alpha_1-\beta_1)] - e^{-\alpha_1-\beta_2}(\alpha_2-\beta_2),$$

where $h_1(u) := \inf_{0<|x|\le u}(e^{-x} - 1 + x)/x^2 > 0$.

Proof. Since $e^{-x} \ge 1 - x$ for $x \in \mathbb{R}$, we have

$$\begin{aligned} e^{-\alpha_1-\alpha_2} - e^{-\beta_1-\beta_2} &= e^{-\beta_2}[e^{-\alpha_1}e^{-(\alpha_2-\beta_2)} - e^{-\beta_1}] \\ &\ge e^{-\beta_2}[e^{-\alpha_1}(1-(\alpha_2-\beta_2)) - e^{-\beta_1}] \\ &= e^{-\beta_1-\beta_2}[e^{-(\alpha_1-\beta_1)} - 1] - e^{-\alpha_1-\beta_2}(\alpha_2-\beta_2) \\ &\ge e^{-\beta_1-\beta_2}[h_1(u)(\alpha_1-\beta_1)^2 - (\alpha_1-\beta_1)] - e^{-\alpha_1-\beta_2}(\alpha_2-\beta_2), \end{aligned}$$

where the last inequality follows from the definition of $h_1(u)$ and $|\alpha_1 - \beta_1| \le u$, completing the proof. □

Lemma 2. *For two measurable subsets S_1 and S_2 of $\mathbb{R}^d$, let $\alpha_1 = \rho\mu_d(C_d \cap S_1)$, $\alpha_2 = \rho\mu_d(C_d \cap S_2)$ and $\beta_2 = \mathrm{E}[\alpha_2] = \rho\mathrm{E}[\mu_d(C_d \cap S_2)]$. Then $\mathrm{E}[e^{-\alpha_1}(\alpha_2 - \beta_2)] \le 0$, i.e. $e^{-\alpha_1}$ and α_2 are nonpositively correlated.*

Proof. For any integrable random variables X and Y with $E|XY| < \infty$,

$$\mathrm{E}[(X-EX)(Y-EY)] = \mathrm{E}[X(Y-EY)] = \mathrm{E}[(X-EX)Y].$$

Noting that $\alpha_2 = \rho\int_{S_2}\mathbf{1}_{C_d}(\mathbf{x})d\mathbf{x}$ where $\mathbf{1}_{C_d}$ denotes the indicator function of C_d, we have

$$\begin{aligned} \mathrm{E}[e^{-\alpha_1}(\alpha_2-\beta_2)] &= \mathrm{E}[(e^{-\alpha_1} - \mathrm{E}[e^{-\alpha_1}])\alpha_2] \\ &= \mathrm{E}\left[(e^{-\alpha_1} - \mathrm{E}[e^{-\alpha_1}])\rho\int_{S_2}\mathbf{1}_{C_d}(\mathbf{x})d\mathbf{x}\right] \\ &= \rho\int_{S_2}\mathrm{E}[(e^{-\alpha_1} - \mathrm{E}[e^{-\alpha_1}])\mathbf{1}_{C_d}(\mathbf{x})]d\mathbf{x} \\ &= \rho\int_{S_2}P(\mathbf{x}\in C_d)\mathrm{E}[e^{-\alpha_1} - \mathrm{E}[e^{-\alpha_1}] \mid \mathbf{x}\in C_d]d\mathbf{x} \\ &= \rho\int_{S_2}P(\Pi\cap B(\mathbf{x}) = \emptyset)\{\mathrm{E}[e^{-\alpha_1} \mid \Pi\cap B(\mathbf{x}) = \emptyset] - \mathrm{E}[e^{-\alpha_1}]\}d\mathbf{x} \end{aligned} \tag{5}$$

where the last equality follows from the fact that $\mathbf{x} \in C_d$ if and only if $\Pi \cap B(\mathbf{x}) = \emptyset$, with $B(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{y}-\mathbf{x}\|_d \le \|\mathbf{x}\|_d\}$ (the ball of radius $\|\mathbf{x}\|_d$ centered at $\mathbf{x}$). We claim that the conditional distribution of $\mu_d(C_d \cap S_1)$ given $\Pi \cap B(\mathbf{x}) = \emptyset$ is stochastically larger than the (unconditional) distribution of $\mu_d(C_d \cap S_1)$. To show this, we make use of the following simple coupling argument. Note by the independence properties of the Poisson process that the conditional distribution of $\mu_d(C_d \cap S_1)$ given $\Pi \cap B(\mathbf{x}) = \emptyset$ is the same as the (unconditional) distribution of $\mu_d(C^* \cap S_1)$ where C^* denotes the Voronoi cell centered at $\mathbf{0}$ generated by $(\Pi \setminus B(\mathbf{x})) \cup \{\mathbf{0}\}$. Since C_d and C^* are defined on the same probability space, we have $C_d \subset C^*$ for every realization of Π, so that $\mu_d(C_d \cap S_1) \le \mu_d(C^* \cap S_1)$ with probability 1. It follows that the (unconditional) distribution of $\mu_d(C_d \cap S_1)$ is stochastically smaller than the conditional distribution of $\mu_d(C_d \cap S_1)$ given $\Pi \cap B(\mathbf{x}) = \emptyset$. Since $e^{-\alpha_1} = f(\mu_d(C_d \cap S_1))$ with $f(x) = e^{-\rho x}$ (a decreasing function), we have

$$\mathrm{E}[e^{-\alpha_1} \mid \Pi\cap B(\mathbf{x}) = \emptyset] \le \mathrm{E}[e^{-\alpha_1}],$$

which together with (5) implies that $\mathrm{E}[e^{-\alpha_1}(\alpha_2 - \beta_2)] \le 0$. The proof is complete. □

Remark 2. It is shown in Yao (2010) that for any measurable subsets S_1 and S_2 of $\mathbb{R}^d$, $\mu_d(C_d \cap S_1)$ and $\mu_d(C_d \cap S_2)$ are nonnegatively correlated. We can use the same argument as in the proof of Lemma 2 to show more generally that for any nondecreasing function $f : \mathbb{R} \to \mathbb{R}$, $f(\mu_d(C_d \cap S_1))$ and $\mu_d(C_d \cap S_2)$ are nonnegatively correlated provided that $\mathrm{E}[|f(\mu_d(C_d \cap S_1))|] < \infty$ and $\mathrm{E}[|f(\mu_d(C_d \cap S_1))|\mu_d(C_d \cap S_2)] < \infty$.

Lemma 3. *Assume $\lambda = 1$ (the intensity of Π). Then there exist constants $c > 0$ and $u_0 < \infty$ such that for all $u \ge u_0$ and $1 \le d < \infty$,*

$$\mathrm{var}(\mu_d(C_d \cap B_{u,d})) \ge c\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d,$$

where $B_{u,d} \subset \mathbb{R}^d$ denotes the ball of volume u centered at $\mathbf{0}$.

Proof. We need several results in Alishahi and Sharifitabar (2008, Sections 3 and 4). By Alishahi and Sharifitabar (2008, Remark 3.2), there exists a constant $c' > 0$ such that

$$\mathrm{var}(\mu_d(C_d)) \geq c' \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } 1 \leq d < \infty. \tag{6}$$

Define $R_d(u) = \mu_d^{d-1}(C_d \cap \partial B_{u,d})/\mu_d^{d-1}(\partial B_{u,d})$ where μ_d^{d-1} denotes the $(d-1)$-dimensional Hausdorff measure (surface area) in $\mathbb{R}^d$ and $\partial B_{u,d}$ denotes the boundary of $B_{u,d}$ (which is a $(d-1)$-dimensional sphere). By Alishahi and Sharifitabar (2008, Lemmas 4.1 and 4.2),

$$\mu_d(C_d \setminus B_{u,d}) = \mu_d(C_d) - \mu_d(C_d \cap B_{u,d}) = \int_u^\infty R_d(v)dv,$$

$$\mathrm{E}[R_d(v)] = e^{-v}, \quad \text{and} \quad \mathrm{E}[\mu_d(C_d \setminus B_{u,d})] = \int_u^\infty e^{-v}dv,$$

so that

$$\begin{aligned}
\mathrm{var}(\mu_d(C_d \setminus B_{u,d})) &= \mathrm{E}\left[\int_u^\infty (R_d(v) - e^{-v})dv\right]^2 \\
&\leq \mathrm{E}\left[\int_u^\infty v^2(R_d(v) - e^{-v})^2 dv \int_u^\infty v^{-2}dv\right] \\
&= u^{-1}\int_u^\infty v^2 \mathrm{var}(R_d(v))dv \\
&\leq u^{-1}\int_u^\infty c'' v^3 e^{-v} \frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d dv \\
&= c''\left(u^{-1}\int_u^\infty v^3 e^{-v}dv\right)\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d
\end{aligned} \tag{7}$$

where the Cauchy–Schwarz inequality is applied to the inner (dv) integral to yield the first inequality and the second inequality follows from the fact that

$$\mathrm{var}(R_d(v)) \leq c'' v e^{-v} \frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } 1 \leq d < \infty$$

for some constant $c'' > 0$ (cf. Alishahi and Sharifitabar, 2008, p. 929). Since $\mu_d(C_d \cap B_{u,d}) + \mu_d(C_d \setminus B_{u,d}) = \mu_d(C_d)$, it follows from Minkowski's inequality, (6) and (7) that for all $1 \leq d < \infty$,

$$\begin{aligned}
\sqrt{\mathrm{var}(\mu_d(C_d \cap B_{u,d}))} &\geq \sqrt{\mathrm{var}(\mu_d(C_d))} - \sqrt{\mathrm{var}(\mu_d(C_d \setminus B_{u,d}))} \\
&\geq \left\{(c')^{1/2} - \left(c'' u^{-1}\int_u^\infty v^3 e^{-v}dv\right)^{1/2}\right\}\left\{\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d\right\}^{1/2}.
\end{aligned}$$

So for all (large) u satisfying $c' > c'' u^{-1}\int_u^\infty v^3 e^{-v}dv$, we have

$$\mathrm{var}(\mu_d(C_d \cap B_{u,d})) \geq h_2(u)\frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } 1 \leq d < \infty,$$

where $h_2(u) = [(c')^{1/2} - (c'' u^{-1}\int_u^\infty v^3 e^{-v}dv)^{1/2}]^2$. Observing that $h_2(u) \to c' > 0$ as $u \to \infty$, the proof is complete. □

Remark 3. According to Alishahi and Sharifitabar (2008, Remark 3.1), the constant c_2 in (4) may be taken to be 5. Since $1 - e^{-\rho} \leq \rho$, (4) may be rewritten as

$$c_1(\rho_0)\rho^2 \frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \leq d_{TV}(\mathcal{L}(N_{\rho,d}), Po(\rho)) \leq c_2\rho^2 \frac{1}{\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d$$

for all $1 \leq d < \infty$ and $0 < \rho \leq \rho_0$. This indicates that $\rho^2 d^{-1/2}(4/3\sqrt{3})^d$ is the rate of Poisson approximation for small ρ and large d.

Remark 4. Theorem 1 indicates that the rate of Poisson approximation for $N_{\rho,d}$ is closely related to the asymptotic behavior of $\mathrm{var}(\mu_d(C_d))$ as $d \to \infty$. While Newman et al. (1983) showed that M_d converges in distribution to $Po(1)$, it remains an open problem to determine the rate of convergence for M_d. More generally, let $M_{d,k}$ be the number of Π-points which have

0 (the typical point of Π) as their kth nearest Π-neighbor, and $N_{\rho,d,k}$ the number of Π'-points which have **0** as their kth nearest Π-neighbor, $k = 1, 2, \ldots$. Note that $M_{d,1} = M_d$ and $N_{\rho,d,1} = N_{\rho,d}$. It is shown in Yao and Simons (1996, Theorem 1) and Yao (2010, Remark 11) that for all k, as $d \to \infty$, $M_{d,k}$ and $N_{\rho,d,k}$ converge in distribution to $Po(1)$ and $Po(\rho)$, respectively. It will also be of interest to determine the rates of convergence for $k \geq 2$.

Remark 5. Consider $m+1$ independent homogeneous Poisson point processes in $\mathbb{R}^d$, Π of intensity λ, Π_i of intensity λ_i, $i = 1, \ldots, m$. Again referring to **0** as the typical point of Π, let $N^i_{\boldsymbol{\rho},d}$ be the number of Π_i-points which have **0** as their nearest Π-neighbor, $i = 1, \ldots, m$, where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m) = (\lambda_1/\lambda, \ldots, \lambda_m/\lambda)$. Let $\Pi' = \cup_{i=1}^m \Pi_i$ and $N_{\rho_{total},d} = \sum_{i=1}^m N^i_{\boldsymbol{\rho},d}$, where $\rho_{total} = \sum_{i=1}^m \rho_i$. Then Π' is a homogeneous Poisson point process of intensity $\sum_{i=1}^m \lambda_i$, independent of Π, and $N_{\rho_{total},d}$ is the number of Π'-points which have **0** as their nearest Π-neighbor. By (4), we have

$$c_1(\rho_0)\rho_{total}^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \leq d_{TV}(\mathcal{L}(N_{\rho_{total},d}), Po(\rho_{total}))$$

$$\leq c_2\rho_{total}(1 - e^{-\rho_{total}}) \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \tag{8}$$

for all $1 \leq d < \infty$ and $0 < \rho_{total} \leq \rho_0$. Since each Π'-point is a Π_i-point with probability ρ_i/ρ_{total}, the conditional distribution of $(N^1_{\boldsymbol{\rho},d}, \ldots, N^m_{\boldsymbol{\rho},d})$ given $N_{\rho_{total},d} = n$ is multinomial with parameters n, ρ_i/ρ_{total}, $i = 1, \ldots, m$. Denote by $Z^1_{\boldsymbol{\rho}}, \ldots, Z^m_{\boldsymbol{\rho}}$, m independent Poisson random variables with respective means $\rho_1, \ldots, \rho_m$. Let $Z_{\rho_{total}} = Z^1_{\boldsymbol{\rho}} + \cdots + Z^m_{\boldsymbol{\rho}}$, which is $Po(\rho_{total})$. Since the conditional distribution of $(Z^1_{\boldsymbol{\rho}}, \ldots, Z^m_{\boldsymbol{\rho}})$ given $Z_{\rho_{total}} = n$ is also multinomial with parameters n, ρ_i/ρ_{total}, $i = 1, \ldots, m$, it follows that

$$P(N^i_{\boldsymbol{\rho},d} = j_i, i = 1, \ldots, m) - P(Z^i_{\boldsymbol{\rho}} = j_i, i = 1, \ldots, m)$$

$$= [P(N_{\rho_{total},d} = n) - P(Z_{\rho_{total}} = n)] \binom{n}{j_1 \cdots j_m} \frac{\rho_1^{j_1} \cdots \rho_m^{j_m}}{\rho_{total}^n}$$

for all $(j_1, \ldots, j_m) \in A_n = \{(\ell_1, \ldots, \ell_m) : \ell_i \geq 0, \ell_1 + \cdots + \ell_m = n\}$, implying that

$$\sum_{(j_1,\ldots,j_m)\in A_n} |P(N^i_{\boldsymbol{\rho},d} = j_i, i = 1, \ldots, m) - P(Z^i_{\boldsymbol{\rho}} = j_i, i = 1, \ldots, m)| = |P(N_{\rho_{total},d} = n) - P(Z_{\rho_{total}} = n)|.$$

So,

$$d_{TV}(\mathcal{L}(N^1_{\boldsymbol{\rho},d}, \ldots, N^m_{\boldsymbol{\rho},d}), \mathcal{L}(Z^1_{\boldsymbol{\rho}}, \ldots, Z^m_{\boldsymbol{\rho}})) = d_{TV}(\mathcal{L}(N_{\rho_{total},d}), \mathcal{L}(Z_{\rho_{total}})),$$

which together with (8) yields the following corollary.

Corollary 1. *For any $\rho_0 > 0$, there exists a constant $c_1(\rho_0) > 0$ such that*

$$c_1(\rho_0)\rho_{total}^2 \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d \leq d_{TV}(\mathcal{L}(N^1_{\boldsymbol{\rho},d}, \ldots, N^m_{\boldsymbol{\rho},d}), Po(\rho_1) \times \cdots \times Po(\rho_m))$$

$$\leq c_2\rho_{total}(1 - e^{-\rho_{total}}) \frac{1}{\sqrt{d}} \left(\frac{4}{3\sqrt{3}}\right)^d,$$

for all $1 \leq d < \infty$ and $0 < \rho_{total} \leq \rho_0$.

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