

## A DUALITY RELATION BETWEEN WORKLOAD AND ATTAINED WAITING TIME IN FCFS $G/G/s$ QUEUES

YI-CHING YAO,\* *Academia Sinica and National Chengchi University*

### Abstract

Sengupta (1989) showed that for the first-come-first-served (FCFS)  $G/G/1$  queue, the workload and attained waiting time of a customer in service have the same stationary distribution. Sakasegawa and Wolff (1990) derived a sample-path version of this result, showing that the empirical distribution of the workload values over a busy period of a given sample path is identical to that of the attained waiting time values over the same period. For a given sample path of an FCFS  $G/G/s$  queue, we construct a dual sample path of a dual queue which is FCFS  $G/G/s$  in reverse time. It is shown that the workload process on the original sample path is identical to the total attained waiting time process on the dual sample path. As an application of this duality relation, we show that for a time-stationary FCFS  $M/M/s/k$  queue, the workload process is equal in distribution to the time-reversed total attained waiting time process.

*Keywords:* Duality; workload process; sample path; reverse time

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Sengupta [3] showed that for the first-come-first-served (FCFS)  $G/G/1$  queue, the workload and attained waiting time of a customer in service have the same stationary distribution. Sakasegawa and Wolff [2] derived a sample-path version of this result, showing that the empirical distribution of the workload values over a busy period of a given sample path is identical to that of the attained waiting time values over the same period. See also Cohen [1] for a different proof and Toyoizumi [4] for related results. For a general multi-server queueing system, Yamazaki and Miyazawa [5] established that the workload and total

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\* Postal address: Institute of Statistical Science, Academia Sinica, Taipei 115 Taiwan, ROC; E-mail address: yao@stat.sinica.edu.tw

attained waiting time of customers in service are identical in average, but their distributions are unequal in general. We complement this result by deriving a duality relation between the workload process and the total attained waiting time process (in reverse time) in FCFS  $G/G/s$  queues. As an application of this duality relation, we show that for a time-stationary FCFS  $M/M/s/k$  queue, the workload process is equal in distribution to the time-reversed total attained waiting time process.

Specifically, consider a sample path  $\omega$  of an FCFS  $G/G/s$  queue. We will construct a corresponding sample path  $\omega^*$  of a dual queue which is FCFS  $G/G/s$  in reverse time, and show that the workload process on  $\omega$  is identical to the total attained waiting time process on  $\omega^*$ . Let  $[\underline{t}, \bar{t}]$  be a busy period of  $\omega$  during which  $n$  customers  $C_1, \dots, C_n$  enter the system with corresponding arrival and departure times,  $A_i < D_i$ ,  $i = 1, \dots, n$ , where  $\underline{t} = A_1 < A_2 < \dots < A_n < \bar{t}$  (implying no batch arrivals). We also assume that the departure times  $D_1, \dots, D_n$  are all distinct. Let  $\pi$  be the permutation of  $(1 \ 2 \ \dots \ n)$  such that

$$\underline{t} < D_{\pi(1)} < D_{\pi(2)} < \dots < D_{\pi(n)} = \bar{t}. \quad (1)$$

To be consistent with the assumption of FCFS service discipline,  $\pi$  needs to satisfy the requirement that

$$\pi(i) < s + i \text{ for all } 1 \leq i \leq n. \quad (2)$$

In words, the  $i$ -th departing customer must be one of  $C_1, \dots, C_{s+i-1}$ . Note that (2) is equivalent to

$$j > \pi(i) \text{ for all } 1 \leq i, j \leq n \text{ with } i \leq j - s. \quad (3)$$

The service time of each customer can be determined as follows. Each of  $C_1, \dots, C_s$  begins service upon arrival, so that  $C_i$ 's service time  $S_i = D_i - A_i$ ,  $i = 1, \dots, s$ . Customer  $C_{s+1}$  begins service either upon arrival if at least one of  $C_1, \dots, C_s$  has already completed service, or at the first departure time  $D_{\pi(1)}$  otherwise. More

generally,  $C_{s+i}$  begins service at  $\max\{A_{s+i}, D_{\pi(i)}\}$ , so that

$$S_{s+i} = D_{s+i} - \max\{A_{s+i}, D_{\pi(i)}\}, i = 1, \dots, n - s. \quad (4)$$

The workload  $V(t)$  at time  $t \in [\underline{t}, \bar{t}]$  is the total remaining service time of customers in the system at time  $t$  (which is also known as the virtual waiting time for the case  $s = 1$ ). The arrival and departure times  $A_i, i = 2, \dots, n, D_{\pi(i)}, i = 1, \dots, n - 1$  divide the interval  $[\underline{t}, \bar{t}] = [A_1, D_{\pi(n)}]$  into  $2n - 1$  subintervals. In each (open) subinterval,  $V(t)$  is linear with a slope  $\in \{-1, \dots, -s\}$ . (More precisely, the absolute value of the slope equals the smaller of  $s$  and the number of customers in the system during this subperiod.) Let  $L(t) = |\{i : A_i < t < D_i\}|$ , the number of customers in the system at  $t \notin \{A_j, D_j, j = 1, \dots, n\}$ , which is constant in each of the  $2n - 1$  (open) subintervals. The following lemma can be proved easily.

**Lemma 1.** *Assume  $n \geq 2$ . Let  $(t_1, t_2)$  and  $(t_2, t_3)$  be two consecutive subintervals ( $t_1 < t_2 < t_3$ ).*

(i) *If  $t_2 = D_i$  for some  $i$ , then  $V(t)$  is continuous at  $t_2$  and  $V'(t_2+) - V'(t_2-) = 0$  or  $1$  according to  $L(t_2-) > s$  or  $\leq s$ .*

(ii) *If  $t_2 = A_i$  for some  $i$ , then  $V(t_2+) - V(t_2-) = S_i$  (the service time of  $C_i$ ) and  $V'(t_2+) - V'(t_2-) = -1$  or  $0$  according to  $L(t_2-) < s$  or  $\geq s$ .*

(iii)  *$V(t) = \bar{t} - t$  in the last subinterval  $(D_{\pi(n-1)}, \bar{t})$ .*

Let  $W_{a,i}(t)$  denote the attained waiting time of  $C_i$  at  $t$ , which is defined as  $t - A_i$  if  $C_i$  is in service at  $t$  or  $0$  otherwise. Let  $W_a(t) = \sum_{i=1}^n W_{a,i}(t)$ , the total attained waiting time of all customers in service at  $t$ , which is linear in each of the  $2n - 1$  subintervals with a slope  $\in \{1, 2, \dots, s\}$  (depending on the number of customers in service). Lemma 2 can also be proved easily.

**Lemma 2.** *Assume  $n \geq 2$ . Let  $(t_1, t_2)$  and  $(t_2, t_3)$  be two consecutive subintervals ( $t_1 < t_2 < t_3$ ).*

(i) Suppose  $t_2 = A_i$  for some  $i$ . Then  $W_a(t)$  is continuous at  $t_2$  and  $W'_a(t_2+) - W'_a(t_2-) = 1$  or  $0$  according to  $L(t_2-) < s$  or  $\geq s$ .

(ii) Suppose  $t_2 = D_i$  for some  $i$ . If  $L(t_2-) > s$ , some customer  $C_j$  begins service at  $t_2$ , and then  $W_a(t_2+) - W_a(t_2-) = W_{a,j}(t_2+) - W_{a,i}(t_2-) = A_i - A_j$  and  $W'_a(t_2+) - W'_a(t_2-) = 0$ . If  $L(t_2-) \leq s$ , then  $W_a(t_2+) - W_a(t_2-) = -W_{a,i}(t_2-) = A_i - D_i = A_i - t_2$  and  $W'_a(t_2+) - W'_a(t_2-) = -1$ .

(iii)  $W_a(t) = t - \underline{t}$  in the first subinterval  $(\underline{t}, A_2)$ .

We now construct a (time-reversed) sample path  $\omega^*$  of a dual FCFS  $G/G/s$  queue in reverse time which has the same busy periods as  $\omega$ . In the busy period  $[\underline{t}, \bar{t}]$  of  $\omega^*$ ,  $n$  customers  $C_1^*, \dots, C_n^*$  enter the system with corresponding arrival and departure times  $A_i^* > D_i^*$ ,  $i = 1, \dots, n$ , where  $A_i^* = D_i$  and  $D_i^* = A_i$ . Since  $\underline{t} < D_{\pi(1)} < \dots < D_{\pi(n)} = \bar{t}$ , we have  $\underline{t} < A_{\pi(1)}^* < \dots < A_{\pi(n)}^* = \bar{t}$ , implying that the  $n$  customers arrive in the order  $C_{\pi(n)}^*, C_{\pi(n-1)}^*, \dots, C_{\pi(1)}^*$  (since these customers arrive in reverse-time order). Since  $\underline{t} = A_1 < A_2 < \dots < A_n < \bar{t}$ , we have  $\underline{t} = D_1^* < D_2^* < \dots < D_n^* < \bar{t}$ , implying that the  $n$  customers leave the system in the order  $C_n^*, C_{n-1}^*, \dots, C_1^*$ . By (3), we have  $j \in \{\pi(n), \pi(n-1), \dots, \pi(j-s+1)\}$ , i.e. the  $(n-j+1)$ -th departing customer must be one of the first  $n-j+s$  arriving customers, implying that the arrival and departure times  $A_i^*, D_i^*$ ,  $i = 1, \dots, n$ , are consistent with the requirement of FCFS service discipline. (Note that the  $j$ -th arriving customer in forward time corresponds to the  $(n-j+1)$ -th departing customer in reverse time.) So  $\omega^*$  is a valid sample path of a (dual) FCFS  $G/G/s$  queue in reverse time.

Let  $W_{a,i}^*(t)$  be the attained waiting time of  $C_i^*$  at  $t$ , i.e.  $W_{a,i}^*(t) = A_i^* - t$  if  $C_i^*$  is in service at  $t$  and  $0$  otherwise. Let  $W_a^*(t) = \sum_{i=1}^n W_{a,i}^*(t)$ , the total attained waiting of all customers in service at  $t$  for  $\omega^*$ . Note that the  $2n-2$  points in  $\{A_i^*, D_i^*, i = 1, \dots, n\} \setminus \{D_1^*, A_{\pi(n)}^*\} = \{A_i, D_i, i = 1, \dots, n\} \setminus \{A_1, D_{\pi(n)}\}$  divide  $[\underline{t}, \bar{t}]$  into  $2n-1$  subintervals. In each (open) subinterval,  $W_a^*(t)$  is linear with a slope  $\in \{-1, \dots, -s\}$ . Since the definitions of  $V, W_a$  and  $W_a^*$  at arrival/departure

times may be ambiguous, we take the convention that  $V$  and  $W_a^*$  are right-continuous and  $W_a$  is left-continuous. Note also that  $L(t) = |\{i : A_i < t < D_i\}| = |\{i : D_i^* < t < A_i^*\}|$  so that the number of customers in the system at  $t$  for  $\omega$  is the same as that for  $\omega^*$ .

Lemma 2 is stated in terms of  $\omega$ . To facilitate the proof of Theorem 1 below, it is convenient to rewrite Lemma 2 in terms of  $\omega^*$  as follows.

**Lemma 2'.** *Assume  $n \geq 2$ . Let  $(t_1, t_2)$  and  $(t_2, t_3)$  be two consecutive subintervals ( $t_1 < t_2 < t_3$ ).*

- (i) *Suppose  $t_2 = A_i^*$  for some  $i$ . Then  $W_a^*(t)$  is continuous at  $t_2$  and  $W_a^{*'}(t_2-) - W_a^{*'}(t_2+) = -1$  or  $0$  according to  $L(t_2+) < s$  or  $\geq s$ .*
- (ii) *Suppose  $t_2 = D_i^*$  for some  $i$ . If  $L(t_2+) > s$ , some customer  $C_j^*$  begins service at  $t_2$ , and then  $W_a^*(t_2-) - W_a^*(t_2+) = W_{a,j}^*(t_2-) - W_{a,i}^*(t_2+) = (A_j^* - t_2) - (A_i^* - t_2) = A_j^* - A_i^*$  and  $W_a^{*'}(t_2-) - W_a^{*'}(t_2+) = 0$ . If  $L(t_2+) \leq s$ , then  $W_a^*(t_2-) - W_a^*(t_2+) = -W_{a,i}^*(t_2+) = t_2 - A_i^* = D_i^* - A_i^*$  and  $W_a^{*'}(t_2-) - W_a^{*'}(t_2+) = 1$ .*
- (iii)  *$W_a^*(t) = \bar{t} - t$  in the “first” subinterval  $(A_{\pi(n-1)}^*, \bar{t})$ .*

**Theorem 1.** *In the busy period  $[\underline{t}, \bar{t}]$ , we have  $V(t) = W_a^*(t)$  for all  $t$ .*

Before proving the theorem, we illustrate the result with Figure 1. We assume there are two servers ( $s = 2$ ) and four customers enter the system during the busy period  $[\underline{t}, \bar{t}] = [0, 7]$  with  $A_1 = D_1^* = 0$ ,  $A_2 = D_2^* = 1$ ,  $A_3 = D_3^* = 2$ ,  $A_4 = D_4^* = 4$ ,  $D_1 = A_1^* = 5$ ,  $D_2 = A_2^* = 3$ ,  $D_3 = A_3^* = 7$ ,  $D_4 = A_4^* = 6$ . Note that this implies  $S_1 = D_1 - A_1 = 5$ ,  $S_2 = D_2 - A_2 = 2$ ,  $S_3 = D_3 - D_2 = 4$ ,  $S_4 = D_4 - D_1 = 1$ , and that customers  $C_3^*, C_4^*, C_1^*, C_2^*$  enter the system at times 7, 6, 5, 3 (in reverse time) and begin service at times 7, 6, 4, 2, respectively.

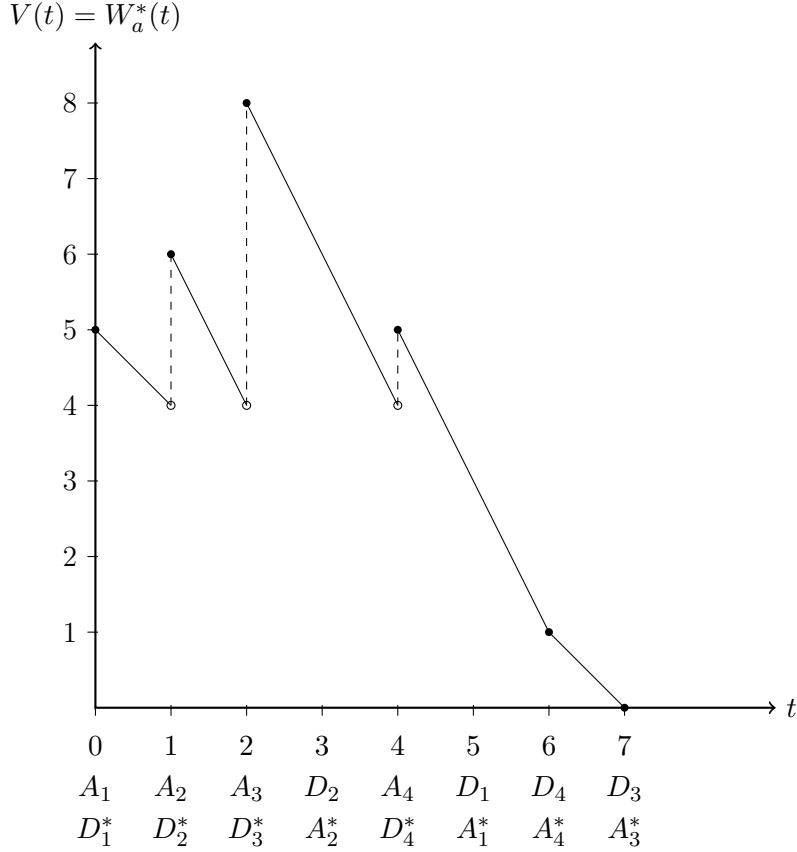


Figure 1

*Proof of Theorem 1.* The case  $n = 1$  is trivial. Assume  $n \geq 2$ . The  $2n - 2$  points in  $\{A_i, D_i, i = 1, \dots, n\} \setminus \{A_1, D_{\pi(n)}\} = \{A_i^*, D_i^*, i = 1, \dots, n\} \setminus \{D_1^*, A_{\pi(n)}^*\}$  divide  $[\underline{t}, \bar{t}]$  into  $2n - 1$  subintervals. In the subinterval  $(D_{\pi(n-1)}, \bar{t}) = (A_{\pi(n-1)}^*, \bar{t})$ , which is the last subinterval for  $\omega$  and the “first” subinterval for  $\omega^*$ , we have  $V(t) = \bar{t} - t = W_a^*(t)$  by Lemmas 1(iii) and 2'(iii). Consider two consecutive subintervals  $(t_1, t_2)$  and  $(t_2, t_3)$ ,  $t_1 < t_2 < t_3$ . Suppose  $V(t) = W_a^*(t)$  for  $t \in (t_2, t_3)$ . We will show that  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ , which by induction yields that  $V(t) = W_a^*(t)$  in each of the  $2n - 1$  open subintervals. It then follows from the right-continuity of  $V$  and  $W_a^*$  that  $V(t) = W_a^*(t)$  for  $t \in [\underline{t}, \bar{t}]$ .

It remains to show  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ . We need to consider the

following cases separately.

(i) Suppose  $t_2 = D_i = A_i^*$  for some  $i$ . By Lemmas 1(i) and 2'(i), we have  $V(t_2-) = V(t_2+) = W_a^*(t_2+) = W_a^*(t_2-)$  where the second equality follows from the induction hypothesis that  $V(t) = W_a^*(t)$  for  $t \in (t_2, t_3)$ .

Two subcases are considered below.

(i.1) Suppose  $L(t_2-) \leq s$  (implying  $L(t_2+) < s$ ). By Lemmas 1(i) and 2'(i),  $V'(t_2+) - V'(t_2-) = 1$  and  $W_a^{*'}(t_2-) - W_a^{*'}(t_2+) = -1$ , implying  $V'(t_2-) = W_a^{*'}(t_2-)$  since  $V'(t_2+) = W_a^{*'}(t_2+)$  by the induction hypothesis. It follows from the linearity of  $V(t)$  and  $W_a^*(t)$  in  $(t_1, t_2)$  that  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ .

(i.2) Suppose  $L(t_2-) > s$  (implying  $L(t_2+) \geq s$ ). By Lemmas 1(i) and 2'(i) and the induction hypothesis,  $V_a'(t_2-) = V_a'(t_2+) = W_a^{*'}(t_2+) = W_a^{*'}(t_2-)$ , implying that  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ .

(ii) Suppose  $t_2 = A_i = D_i^*$  for some  $i$ . Then  $V(t_2+) - V(t_2-) = S_i$ , the service time of  $C_i$ .

(ii.1) Suppose  $L(t_2-) < s$  (implying  $L(t_2+) \leq s$ ). Then  $C_i$  begins service upon arrival, so  $S_i = D_i - A_i$  and  $V(t_2+) - V(t_2-) = D_i - A_i$ . Also  $W_a^*(t_2+) - W_a^*(t_2-) = W_{a,i}^*(t_2+) = A_i^* - D_i^* = D_i - A_i$ , implying  $V(t_2-) = W_a^*(t_2-)$ . Furthermore, by Lemmas 1(ii) and 2'(ii),  $V'(t_2+) - V'(t_2-) = -1$  and  $W_a^{*'}(t_2-) - W_a^{*'}(t_2+) = 1$ , implying  $V'(t_2-) = W_a^{*'}(t_2-)$  since  $V'(t_2+) = W_a^{*'}(t_2+)$  by the induction hypothesis. So  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ .

(ii.2) Suppose  $L(t_2-) \geq s$  (implying  $L(t_2+) > s$ ). By (4), we have

$$\begin{aligned} V(t_2+) - V(t_2-) &= S_i = D_i - \max\{A_i, D_{\pi(i-s)}\} \\ &= D_i - D_{\pi(i-s)}, \end{aligned} \tag{5}$$

since  $C_i$  begins service at  $\max\{A_i, D_{\pi(i-s)}\} = D_{\pi(i-s)}$ . As for  $\omega^*$ ,  $C_i^*$  completes service at  $t_2 = D_i^*$  and another customer begins service at  $t_2$

since  $L(t_2+) > s$ . By time  $t_2 = D_i^*(= A_i)$ , the number of customers who have completed service is  $n - i + 1$ , since the departure times before  $t_2$  (in reverse time) are  $D_{i+1}^* = A_{i+1}, \dots, D_n^* = A_n$ . Thus the customer who begins service at  $t_2$  must be the  $(n - i + 1 + s)$ -th customer entering the system, i.e.  $C_{\pi(i-s)}^*$ . So

$$\begin{aligned} W_a^*(t_2-) - W_a^*(t_2+) &= W_{a,\pi(i-s)}^*(t_2-) - W_{a,i}^*(t_2+) \\ &= (A_{\pi(i-s)}^* - t_2) - (A_i^* - t_2) \\ &= D_{\pi(i-s)} - D_i, \end{aligned}$$

which together with (5) and the induction hypothesis yields that  $V(t_2-) = W_a^*(t_2-)$ . Finally by Lemmas 1(ii) and 2'(ii) and the induction hypothesis,  $V'(t_2-) = V'(t_2+) = W_a^{*'}(t_2+) = W_a^{*'}(t_2-)$ , implying  $V(t) = W_a^*(t)$  for  $t \in (t_1, t_2)$ . This completes the proof.

**Remark 1.** In the arguments above, we have implicitly assumed that  $A_j \neq D_i$  for all  $i, j$ . This restriction can easily be relaxed as follows: Let  $I = \{i : A_i = D_j \text{ for some } j\}$ . For (sufficiently small)  $\varepsilon > 0$ , define  $A_{k,\varepsilon} = A_k - \varepsilon \mathbf{1}_I(k)$  and  $D_{k,\varepsilon} = D_k$ ,  $k = 1, \dots, n$ , where  $\mathbf{1}_I(k) = 1$  or  $0$  according to  $k \in I$  or  $k \notin I$ . Then by applying Theorem 1 to the arrival and departure times  $\{A_{k,\varepsilon}, D_{k,\varepsilon}, k = 1, \dots, n\}$ , we have  $V_\varepsilon(t) = W_{a,\varepsilon}^*(t)$  for  $t \in [\underline{t}, \bar{t}]$ . Letting  $\varepsilon \searrow 0$  yields  $V(t) = W_a^*(t)$ .

**Remark 2.** A stochastic FCFS  $G/G/s$  queueing system gives rise to a probability measure on the space of sample paths  $\omega$  (endowed with a suitably specified  $\sigma$ -field), which induces a probability measure on the space of time-reversed sample paths  $\omega^*$ , which in turn defines a dual FCFS  $G/G/s$  queue in reverse time. We may also think of both queues as coupled (i.e. defined on the same probability space). Then by Theorem 1,  $V(t) = W_a^*(t)$  for all  $t$  with probability 1 where  $V$  is the workload process for the original queue and  $W_a^*$  is the total attained waiting time process for the dual queue. It would be of interest to characterize the dual queues for some classes of queueing systems. For an FCFS  $M/M/s/k$  queue ( $k$



being the system capacity), it can readily be argued that the dual queue is also FCFS  $M/M/s/k$  in reverse time, which together with Theorem 1 implies that the workload process is equal in distribution to the total attained waiting process in reverse time. This is summarized in the following theorem.

**Theorem 2.** *For a time-stationary FCFS  $M/M/s/k$  queue with  $1 \leq s \leq k \leq \infty$ , the workload process  $V(t)$  is equal in distribution to the time-reversed total attained waiting time process  $W_a(t)$ , i.e.*

$$(V(t_1), \dots, V(t_k)) \stackrel{d}{=} (W_a(-t_1), \dots, W_a(-t_k)) \text{ for all } t_1, \dots, t_k, k \geq 1.$$

Consequently,  $V(t)$  and  $W_a(t)$  have the same stationary distribution and the same autocovariance function, i.e.  $\text{Cov}(V(0), V(t)) = \text{Cov}(W_a(0), W_a(t))$  for all  $t$ .

*Proof of Theorem 2.* For a time-stationary FCFS  $M/M/s/k$  queue, let  $L(t)$  denote the queue length (number of customers in the system) at time  $t$ . As a birth-and-death process,  $L(t)$  is time-reversible. Let  $L(\cdot) = \{L(t) : -\infty < t < \infty\}$ , which is a random sample path such that the value of  $L(\cdot)$  at time  $t$  is  $L(t)$ . Fix a realization  $\omega_L$  of  $L(\cdot)$ . Let  $[\underline{t}, \bar{t}]$  be a busy period of  $\omega_L$  during which  $n$  customers  $C_1, \dots, C_n$  enter the system with corresponding arrival times  $\underline{t} = A_1 < A_2 < \dots < A_n < \bar{t}$ . Denote the ordered departure times by  $\underline{t} < \tilde{D}_1 < \tilde{D}_2 < \dots < \tilde{D}_n = \bar{t}$ . Necessarily,  $|\{i : A_i < \tilde{D}_j\}| > j$  for  $j < n$ . While the information of the  $A_i$ 's and  $\tilde{D}_i$ 's is contained in  $\omega_L$ , the departure time  $D_i$  of  $C_i$  may not be observable given  $\omega_L$ . Since by (1) the (unobservable) permutation  $\pi$  satisfies  $D_{\pi(1)} < \dots < D_{\pi(n)}$ , we have  $D_{\pi(i)} = \tilde{D}_i$  for all  $i$ , i.e.  $D_i = \tilde{D}_{\pi^{-1}(i)}$  for all  $i$ . A permutation  $\phi$  of  $(1 \ 2 \ \dots \ n)$  is said to be an admissible matching if the paired arrival and departure times  $(A_i, \tilde{D}_{\phi^{-1}(i)})$ ,  $i = 1, \dots, n$ , satisfy the requirement of FCFS service discipline (cf. (2)). Since  $(A_i, \tilde{D}_{\pi^{-1}(i)}) = (A_i, D_i)$ , the (unobservable) permutation  $\pi$  is (necessarily) admissible which will be referred to as the true matching. Note that the identity permutation is always an admissible matching. It is not difficult to show that

- (i) given  $L(\cdot) = \omega_L$ , the true matching  $\pi$  for the busy period  $[\underline{t}, \bar{t}]$  is (*conditionally*) equally likely to be any one of the admissible matchings,
- (ii) given  $L(\cdot) = \omega_L$ , the true matchings in different busy periods are (*conditionally*) independent.

(See Remark 3 below for further discussion.)

While  $\omega_L$  consists only of the information of arrival and departure times, let  $\omega$  be an expanded sample path version of  $\omega_L$  which also includes the information of paired arrival and departure times. More precisely, write  $\omega = (\omega_L, \Phi(\omega_L))$  where the second component  $\Phi(\omega_L)$  (a rule matching arrival times with departure times) specifies an admissible matching in each of the busy periods of  $\omega_L$ . Let  $\omega^* = (\omega_L, \Phi(\omega_L))^* = (\omega_L^*, \Phi(\omega_L^*))$  be the corresponding (time-reversed) sample path of a dual queue as defined earlier, where  $\omega_L^*$  is  $\omega_L$  with the roles of arrival times and departure times interchanged and the matching rule  $\Phi(\omega_L^*)$  is such that an arrival time  $t'$  is matched with a departure time  $t''$  under  $\Phi(\omega_L^*)$  if and only if  $t'$  (departure time) and  $t''$  (arrival time) are matched under  $\Phi(\omega_L)$ . Furthermore, let  $L_\Pi = (L(\cdot), \Pi(L(\cdot)))$  be the random (expanded) sample path where the second component  $\Pi(L(\cdot))$  specifies the (unobservable) true matching in each of the busy periods of  $L(\cdot)$ . By (i) and (ii), it is instructive to think of the second component  $\Pi(L(\cdot))$  as if it picks an admissible matching *at random* in each of the busy periods of  $L(\cdot)$ . Consequently, for the random time-reversed (expanded) sample path  $L_\Pi^* = (L(\cdot), \Pi(L(\cdot)))^* = (L^*(\cdot), \Pi(L^*(\cdot)))$ , we may also think of the second component  $\Pi(L^*(\cdot))$  as if it picks an admissible matching *at random* in each of the busy periods of the first component  $L^*(\cdot)$ . Since  $L(\cdot)$  is time-reversible (i.e.  $L^*(\cdot)$  is a time-reversed copy of  $L(\cdot)$ ), it follows that the random (expanded) sample path  $L_\Pi^*$  is a time-reversed copy of the random (expanded) sample path  $L_\Pi$ .

Now let  $V(\cdot)$  be the workload process on the random (expanded) sample path  $L_\Pi$  and let  $W_a(\cdot)$  (or  $W_a^*(\cdot)$ , respectively) be the total attained waiting time process on  $L_\Pi$  (or  $L_\Pi^*$ , respectively). Since  $L_\Pi^*$  is a time-reversed copy of  $L_\Pi$ ,

$W_a^*(\cdot)$  is a time-reversed copy of  $W_a(\cdot)$ , i.e. for all  $t_1, \dots, t_k, u_1, \dots, u_k, k \geq 1$ ,

$$P(W_a^*(t_i) \leq u_i, i = 1, \dots, k) = P(W_a(-t_i) \leq u_i, i = 1, \dots, k).$$

By Theorem 1,

$$P(V(t_i) \leq u_i, i = 1, \dots, k) = P(W_a^*(t_i) \leq u_i, i = 1, \dots, k),$$

showing that  $(V(t_1), \dots, V(t_k))$  and  $(W_a(-t_1), \dots, W_a(-t_k))$  have the same joint distribution. This completes the proof.

**Remark 3.** Denote by  $(a_1 d_1 \cdots a_r d_r)$  the arrival-departure run pattern determined by the (ordered) arrival and departure times  $A_i$ 's and  $\tilde{D}_i$ 's in the busy period  $[t, \bar{t}]$ , i.e.

$$A_1 < \cdots < A_{a_1} < \tilde{D}_1 < \cdots < \tilde{D}_{d_1} < A_{a_1+1} < \cdots < A_{a_1+a_2} < \tilde{D}_{d_1+1} < \cdots < \tilde{D}_{d_1+d_2} < \cdots < A_{a_1+\cdots+a_{r-1}+1} < \cdots < A_n < \tilde{D}_{d_1+\cdots+d_{r-1}+1} < \cdots < \tilde{D}_n.$$

The  $a_i$ 's and  $d_i$ 's need to satisfy

$$\sum_{i=1}^j a_i > \sum_{i=1}^j d_i, j = 1, \dots, r-1; \sum_{i=1}^r a_i = \sum_{i=1}^r d_i = n.$$

The number of admissible matchings is a function of  $(a_1 d_1 \cdots a_r d_r)$ , which will be denoted by  $N_r(a_1 d_1 \cdots a_r d_r)$  and which can be computed recursively as follows. The first  $\min\{a_1, s\}$  arriving customers are equally likely to leave the system first at  $\tilde{D}_1$  since the (common) service time distribution is memoryless. When one of  $A_1, \dots, A_{\min\{a_1, s\}}$  is matched with  $\tilde{D}_1$ , the run pattern reduces to  $((a_1 - 1) (d_1 - 1) \cdots a_r d_r)$  if  $d_1 > 1$  or  $((a_1 + a_2 - 1) d_2 \cdots a_r d_r)$  if  $d_1 = 1$ . This leads to the recursion

$$N_r(a_1 d_1 a_2 d_2 \cdots a_r d_r) = \begin{cases} \min\{a_1, s\} N_r((a_1 - 1) (d_1 - 1) a_2 d_2 \cdots a_r d_r), & \text{if } d_1 > 1; \\ \min\{a_1, s\} N_{r-1}((a_1 + a_2 - 1) d_2 \cdots a_r d_r), & \text{if } d_1 = 1. \end{cases}$$

Alternatively, the departure time  $D_n$  of the last arriving customer  $C_n$  is equally likely to be any one of  $\tilde{D}_i, n - \min\{d_r, s\} + 1 \leq i \leq n$ . When  $A_n$  is matched

with one of these  $\tilde{D}_i$ 's, the run pattern reduces to  $(a_1 d_1 \cdots (a_r - 1) (d_r - 1))$  if  $a_r > 1$  or  $(a_1 d_1 \cdots a_{r-1} (d_{r-1} + d_r - 1))$  if  $a_r = 1$ . This leads to a different but equivalent recursion for  $N_r(a_1 d_1 \cdots a_r d_r)$ .

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