Logconcave reward functions and optimal stopping rules of threshold form

Shoou-Ren Hsiau[*], Yi-Shen Lin[†], and Yi-Ching Yao[‡]

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Abstract

In the literature, the problem of maximizing the expected discounted reward over all stopping rules has been explicitly solved for a number of reward functions (including $(\max\{x, 0\})^\nu$, $\nu > 0$, in particular) when the underlying process is either a random walk in discrete time or a Lévy process in continuous time. All of such reward functions are increasing and logconcave while the corresponding optimal stopping rules have the threshold form. In this paper, we explore the close connection between increasing and logconcave reward functions and optimal stopping rules of threshold form. In the discrete case, we show that if a reward function defined on $\mathbb{Z}$ is nonnegative, increasing and logconcave, then the optimal stopping rule is of threshold form provided the underlying random walk is skip-free to the right. In the continuous case, it is shown that for a reward function defined on $\mathbb{R}$ which is nonnegative, increasing, logconcave and right-continuous, the optimal stopping rule is of threshold form provided the underlying process is a spectrally negative Lévy process. Furthermore, we also establish the necessity of logconcavity and monotonicity of a reward function in order for the optimal stopping rule to be of threshold form in the discrete (continuous, *resp.*) case when the underlying process belongs to the class of Bernoulli random walks (Brownian motions, *resp.*) with a downward drift. These results together provide a partial characterization of the threshold structure of optimal stopping rules.

[*]Department of Mathematics, National Changhua University of Education, No. 1, Jin-De Rd., Changhua 500, Taiwan, R.O.C. Email address: srhsiau@cc.ncue.edu.tw

[†]Department of Mathematics, National Changhua University of Education, No. 1, Jin-De Rd., Changhua 500, Taiwan, R.O.C. Email address: yslin.ysl@gmail.com

[‡]Institute of Statistical Science, Academia Sinica, Taipei 115, Taiwan, R.O.C. Email address: yao@stat.sinica.edu.tw

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1 Introduction

Let $X = \{X_t\}_{t\geq 0}$ be a process with stationary independent increments defined on a probability space $(\Omega, \mathcal{F}, P)$ where the time parameter t is either discrete (i.e. $t \in \mathbb{Z}^+ = \{0, 1, \dots\}$) or continuous (i.e. $t \in \mathbb{R}^+ = [0, \infty)$). We consider the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ where $\mathcal{F}_t$ is the P-completed σ-field generated by $\{X_s : 0 \leq s \leq t\}$. For a given nonnegative measurable reward function g and a discount rate $\gamma \geq 0$, we are concerned with the problem of finding a stopping rule $\tau^* \in \mathcal{M}$ such that

$$E\left(e^{-\gamma\tau^*} g(X_{\tau^*}) \mathbf{1}_{\{\tau^*<\infty\}}\right) = \sup_{\tau\in\mathcal{M}} E\left(e^{-\gamma\tau} g(X_\tau) \mathbf{1}_{\{\tau<\infty\}}\right) ,$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0, \infty]$ (with respect to the filtration $\{\mathcal{F}_t\}$) and $\mathbf{1}_A$ denotes the indicator function of A. A stopping rule τ is of threshold form if $\tau = \inf\{t \geq 0 : X_t \geq a\}$ for some $a \in \mathbb{R} \cup \{-\infty\}$. An optimal stopping rule of threshold form exists if $\tau^* = \tau_a$ for some $a \in \mathbb{R} \cup \{-\infty\}$.

In the literature, Dubins and Teicher [9] solved the problem with $g(x) = x^+ := \max\{x, 0\}$ and $\gamma > 0$ under the discrete-time setting. Darling, Liggett and Taylor [7] also considered $g(x) = x^+$ (with $\gamma = 0$) and $g(x) = (e^x - 1)^+$ both in discrete time, while Mordecki [12] considered $g(x) = (e^x - 1)^+$ and $g(x) = (1 - e^{-x})^+$ in continuous time. Novikov and Shiryaev [13] and Kyprianou and Surya [11] further considered the more general case $g(x) = (x^+)^n$, $n = 1, 2, \dots$ with $\gamma = 0$ in discrete time and with $\gamma \geq 0$ in continuous time, respectively, while [13] also considered $g(x) = 1 - e^{-x^+} = (1 - e^{-x})^+$ with $\gamma = 0$ in discrete time. More recently, by generalizing Appell polynomials to Appell functions, Novikov and Shiryaev [14] were able to extend the results of [11, 13] to the case $g(x) = (x^+)^\nu$ for all real-valued $\nu > 0$ with $\gamma \geq 0$ in both discrete and continuous time. Note that all of the above reward functions are increasing and logconcave, and the corresponding optimal stopping rules obtained in [7, 9, 11, 12, 13, 14] have the threshold form.

To solve the optimal stopping problem for a more general class of reward functions under Lévy processes, Surya [16] introduced an (associated) averaging problem from which a fluctuation identity for overshoots of a Lévy process was obtained. Then the value function and the optimal stopping time can be expressed in terms of the solution to the averaging problem provided this solution exists and has certain monotonicity properties. See also Deligiannidis, Le and Utev [8] for related results on Lévy processes as well as on random walks. More recently, Christensen, Salminen and Ta [6] characterized the solution to the optimal stopping problem similarly as in [8, 16] but under very general strong Markov processes including diffusions, Lévy processes and continuous-time Markov chains. Moreover, the optimal stopping time can be either one-sided or two-sided depending on the form of the representing function for the given reward function. For additional results concerning the threshold structure of optimal stopping rules, see Baurdoux [1] on (generalized) Ornstein-Uhlenbeck processes driven by Lévy processes, and Christensen, Irle and Novikov [5] on AR(1) sequences.

In the present paper, we focus our attention on exploring the close connection between increasing logconcave reward functions and optimal stopping rules of threshold form. Specifically, in Section 2, we consider the case of discrete time and discrete state and show that if a reward function defined on $\mathbb{Z}$ is nonnegative, increasing and logconcave, then the optimal stopping rule is of threshold form provided that the underlying (integer-valued) random walk is skip-free to the right. In Section 3, we treat the continuous case and show that for a reward function defined on $\mathbb{R}$ which is nonnegative, increasing, logconcave and right-continuous, the optimal stopping rule is of threshold form provided that $\{X_t\}_{t\geq 0}$ is a spectrally negative Lévy process. In Sections 4 and 5, we deal with the necessity of logconcavity and monotonicity of a reward function in order for the optimal stopping rule to be of threshold form. Specifically, we consider $\gamma = 0$ (no future discount) and show in Section 4 (Section 5, *resp.*) that a nonnegative reward function defined on $\mathbb{Z}$ ($\mathbb{R}$, *resp.*) is necessarily increasing and logconcave if the corresponding optimal stopping rule is of threshold form for all Bernoulli random walks (Brownian motions, *resp.*) with a downward drift. Section 6 contains concluding remarks.

2 Optimal stopping rules for logconcave reward functions: the discrete case

In this section, we use $n \in \mathbb{Z}^+$ (instead of t) to denote the discrete time parameter. Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed integer-valued random variables such that $P(\xi > 1) = 0$ and $P(\xi = 1) > 0$. Let $T = \inf\{n > 0 : \xi_1 + \xi_2 + \cdots + \xi_n \geq 1\}$ where $\inf \emptyset := \infty$. For $\gamma \geq 0$, define

$$\alpha = \alpha(\gamma) = E\left(e^{-\gamma T}\mathbf{1}_{\{T<\infty\}}\right). \tag{2.1}$$

Here α is defined for all $\gamma \geq 0$, but we need to assume $E(\xi) < 0$ if $\gamma = 0$, so that $0 < \alpha < 1$. By the first-step analysis, α satisfies

$$\alpha = e^{-\gamma} \sum_{\ell \leq 1} P(\xi = \ell)\alpha^{1-\ell}. \tag{2.2}$$

Remark 2.1. *The function*

$$f(x) = e^{-\gamma} \sum_{\ell \leq 1} P(\xi = \ell)x^{1-\ell} - x, \ \ 0 \leq x \leq 1, \tag{2.3}$$

is convex with $f(\alpha) = 0, f(0) = e^{-\gamma}P(\xi = 1) > 0, f(1) = e^{-\gamma} - 1 \leq 0$, *and* $f'(1-) = e^{-\gamma}(1 - E(\xi)) - 1$ *(which is positive if* $\gamma = 0$, *since* $E(\xi) < 0$*), where* $f'(x-)$ *denotes the left-hand derivative of* f *at* x. *Clearly* α *is the unique root of* $f(x) = 0$ *in* $(0, 1)$ *and* $f(x) \leq 0$ *for all* $x \in [\alpha, 1]$.

Let $X_0 = k \in \mathbb{Z}$, $X_{n+1} = X_n + \xi_{n+1}$ for $n = 0, 1, 2, \ldots$, so that $\{X_n\}_{n\geq 0}$ is a random walk with initial state k which is skip-free to the right, i.e. $\{X_n\}_{n\geq 0}$ can only move up one level at a time but can skip down several levels. (See [3] for a discussion of skip-free random walks.) For $\ell \in \mathbb{Z}$, define $\tau_\ell = \inf\{n \geq 0 : X_n \geq \ell\}$ (a stopping rule of threshold form). Then by the skip-free property, we have for $X_0 = k < \ell$ that

$$X_{\tau_\ell} = \ell \ \text{ a.s. on } \{\tau_\ell < \infty\} \text{ and } E_k(e^{-\gamma\tau_\ell}\mathbf{1}_{\{\tau_\ell<\infty\}}) = \alpha^{\ell-k}, \tag{2.4}$$

where the subscript k in E_k refers to the initial state $X_0 = k$. For a (nonnegative) reward function $g : \mathbb{Z} \to [0, \infty)$ which is nonconstant, increasing (i.e. $g(k) \leq g(k + 1)$ for all k) and logconcave (i.e. $(g(k + 1))^2 \geq g(k)g(k + 2)$ for all k), define

$$u = u(\gamma) = \inf\left\{k \in \mathbb{Z} : \frac{g(k)}{g(k+1)} \geq \alpha\right\} \ \ \left(\frac{0}{0} := 0, \ \ \inf \mathbb{Z} := -\infty\right), \tag{2.5}$$

$$U = U(\gamma) = \sup\left\{\alpha^\ell g(\ell) : \ell \in \mathbb{Z}\right\}, \tag{2.6}$$

$$V(k) = V_\gamma(k) = \sup_{\tau\in\mathcal{M}} E_k\left(e^{-\gamma\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right) \ \ (k \in \mathbb{Z}), \tag{2.7}$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0, \infty]$. (Note that $g(k)/g(k + 1)$ is increasing in k since g is logconcave.) We are now ready to state the main result in this section.

Theorem 2.1. *Let $\gamma \geq 0$, and assume $E(\xi) < 0$ if $\gamma = 0$. Let $g : \mathbb{Z} \to [0, \infty)$ be nonconstant, increasing and logconcave, and define $\alpha = \alpha(\gamma), u = u(\gamma), U = U(\gamma)$ and $V(k) = V_\gamma(k)$ as in* (2.1) *and* (2.5) − (2.7)*. Then the following statements hold.*

(i) *If $-\infty < u < \infty$, then the threshold-form stopping rule τ_u is optimal, and $V(k) = g(k)$ for $k \geq u$; and $V(k) = \alpha^{u-k} g(u)$ for $k < u$.*

(ii) *If $u = \infty$, then $V(k) = \alpha^{-k} U$ for all k. If, in addition, $U = \infty$, then there exist (randomized) stopping rules that have an infinite expected (discounted) reward; if $U < \infty$, then there is no optimal stopping rule.*

(iii) *If $u = -\infty$, then $V(k) = g(k)$ for all k and the optimal stopping rule is to stop immediately.*

To prove Theorem 2.1, we need the following standard result (*cf.* [14, Lemma 5]).

Lemma 2.1. *Let $g(k)$ and $h(k)$ be nonnegative functions defined on $\mathbb{Z}$ and $\gamma \geq 0$. If $h(k) \geq g(k)$ and $h(k) \geq E[e^{-\gamma} h(k + \xi)]$ for all k, then*

$$h(k) \geq \sup_{\tau \in \mathcal{M}} E_k \left(e^{-\gamma\tau} g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right), \quad k \in \mathbb{Z}.$$

Proof of Theorem 2.1. (i) Let $\widehat{V}(k) = E_k \left(e^{-\gamma\tau_u} g(X_{\tau_u}) \mathbf{1}_{\{\tau_u < \infty\}} \right), k \in \mathbb{Z}$. Then by (2.4), $\widehat{V}(k) = g(k)$ for $k \geq u$ and $\widehat{V}(k) = \alpha^{u-k} g(u)$ for $k < u$. We need to show $V(k) = \widehat{V}(k)$ for all k. Since clearly $V(k) \geq \widehat{V}(k)$, it remains to prove $\widehat{V}(k) \geq V(k)$. By Lemma 2.1, it suffices to show

$$\widehat{V}(k) > g(k) \text{ for } k < u, \tag{2.8}$$

and

$$\widehat{V}(k) \geq E[e^{-\gamma} \widehat{V}(k + \xi)] \quad \text{for all } k. \tag{2.9}$$

By the definition of u, we have $g(k)/g(k+1) < \alpha$ for $k < u$ and $g(u)/g(u+1) \geq \alpha$, implying that $g(u) > 0$ and $\widehat{V}(k) > 0$ for all k. For $k < u$, $g(k) = \frac{g(k)}{g(k+1)} \frac{g(k+1)}{g(k+2)} \cdots \frac{g(u-1)}{g(u)} g(u) < \alpha^{u-k} g(u) = \widehat{V}(k)$. This proves (2.8).

To prove (2.9), let $h(k) = \alpha^{u-k} g(u)$ for all k. Then for $k > u$,

$$\begin{aligned} \widehat{V}(k) = g(k) &= \frac{g(k)}{g(k-1)} \frac{g(k-1)}{g(k-2)} \cdots \frac{g(u+1)}{g(u)} g(u) \\ &\leq \left(\alpha^{-1} \right)^{k-u} g(u) = h(k). \end{aligned}$$

So $\widehat{V}(k) \le h(k)$ for all k. For $k \le u$,

$$\begin{aligned} E[e^{-\gamma}\widehat{V}(k+\xi)] &\le E[e^{-\gamma}h(k+\xi)] \\ &= e^{-\gamma}\sum_{\ell\le 1} P(\xi=\ell)\alpha^{u-k-\ell}g(u) \\ &= \alpha^{u-k}g(u) = \widehat{V}(k), \end{aligned}$$

where the second equality follows from (2.2).

It remains to prove (2.9) for $k > u$. Noting that $g(k-1) \ge g(u) > 0$ and that $\frac{g(j-1)}{g(j)} \le \frac{g(j)}{g(j+1)}$ for all j, we have

$$\frac{g(k+\ell)}{g(k)} = \frac{g(k+\ell)}{g(k+\ell+1)}\frac{g(k+\ell+1)}{g(k+\ell+2)}\cdots\frac{g(k-1)}{g(k)} \le \left(\frac{g(k-1)}{g(k)}\right)^{-\ell} \quad \text{for } \ell < 0.$$

A similar argument shows that the above inequality also holds for $\ell > 0$; thus,

$$\frac{g(k+\ell)}{g(k)} \le \left(\frac{g(k-1)}{g(k)}\right)^{-\ell} \quad \text{for all } \ell. \tag{2.10}$$

(Note that (2.10), in fact, holds for any k with $g(k-1) > 0$ regardless of whether $k > u$. This inequality is also needed later in the proof of part (iii).) For $k > u$ and $\ell \le u-k$ (i.e. $k+\ell \le u$), we have

$$\begin{aligned} \frac{\widehat{V}(k+\ell)}{\widehat{V}(k)} &= \alpha^{u-k-\ell}\frac{g(u)}{g(k)} \le \alpha^{u-k-\ell}\left(\frac{g(k-1)}{g(k)}\right)^{k-u} \\ &\le \left(\frac{g(k-1)}{g(k)}\right)^{u-k-\ell}\left(\frac{g(k-1)}{g(k)}\right)^{k-u} = \left(\frac{g(k-1)}{g(k)}\right)^{-\ell}, \end{aligned} \tag{2.11}$$

where the first inequality is by (2.10) and the second inequality follows from $\alpha \le g(u)/g(u+1) \le g(k-1)/g(k)$. For $k > u$ and $\ell > u-k$ (i.e. $k+\ell > u$), we have by (2.10)

$$\frac{\widehat{V}(k+\ell)}{\widehat{V}(k)} = \frac{g(k+\ell)}{g(k)} \le \left(\frac{g(k-1)}{g(k)}\right)^{-\ell},$$

which together with (2.11) implies that $\widehat{V}(k+\ell)/\widehat{V}(k) \le (g(k-1)/g(k))^{-\ell}$ for $k > u$ and for all ℓ. So, for $k > u$

$$\begin{aligned} \frac{E[e^{-\gamma}\widehat{V}(k+\xi)]}{\widehat{V}(k)} &= e^{-\gamma}\sum_{\ell\le 1} P(\xi=\ell)\frac{\widehat{V}(k+\ell)}{\widehat{V}(k)} \\ &\le e^{-\gamma}\sum_{\ell\le 1} P(\xi=\ell)\left(\frac{g(k-1)}{g(k)}\right)^{-\ell} = \frac{f(c)}{c} + 1 \le 1, \end{aligned}$$

where the second equality follows from (2.3) with $c := g(k-1)/g(k)$ and the last inequality is due to $f(c) \le 0$ since $\alpha \le g(u)/g(u+1) \le g(k-1)/g(k) = c \le 1$ (*cf.* Remark 2.1). This proves (2.9) and completes the proof of part (i).

(ii) Let $\widehat{V}(k) = \alpha^{-k}U$ for all k. We need to show $V(k) = \widehat{V}(k)$. Note that since $u = \infty$, $\frac{\alpha g(k+1)}{g(k)} > 1$ for all k with $g(k+1) > 0$, so $\alpha^k g(k)$ is strictly increasing in $k \ge k_0 := \inf\{\ell : g(\ell) > 0\}$. Since g is nonconstant, we have $k_0 < \infty$ and $U > \alpha^k g(k)$ for all k, implying that $\widehat{V}(k) > g(k)$ for all k. For $X_0 = k < \ell$, we have by (2.4)

$$\begin{aligned} V(k) &\ge E_k\left(e^{-\gamma\tau_\ell} g(X_{\tau_\ell})\mathbf{1}_{\{\tau_\ell<\infty\}}\right) \\ &= g(\ell)E_k(e^{-\gamma\tau_\ell}\mathbf{1}_{\{\tau_\ell<\infty\}}) \\ &= \alpha^{\ell-k}g(\ell) \to \alpha^{-k}U = \widehat{V}(k) \text{ as } \ell\to\infty, \end{aligned}$$

implying that $V(k) \ge \widehat{V}(k)$ for all k.

Suppose $U = \infty$. Obviously $V(k) = \widehat{V}(k) = \infty$. Choose an increasing sequence of $k_1 < k_2 < \cdots$ such that $\alpha^{k_n} g(k_n) > 2^n$ for all n. Then consider a randomized stopping rule of threshold form which chooses threshold k_n with probability $\frac{1}{2^n}$. Clearly this stopping rule has an infinite expected (discounted) reward.

Suppose $U < \infty$. We claim $\widehat{V}(k) = E[e^{-\gamma}\widehat{V}(k+\xi)]$, i.e. $\{e^{-\gamma n}\widehat{V}(X_n)\}_{n\ge0}$ is a martingale. To establish this claim, note that

$$\begin{aligned} E[e^{-\gamma}\widehat{V}(k+\xi)] &= \sum_{\ell\le1} P(\xi=\ell)e^{-\gamma}\widehat{V}(k+\ell) \\ &= e^{-\gamma}\sum_{\ell\le1} P(\xi=\ell)\alpha^{-k-\ell}U \\ &= \left(e^{-\gamma}\sum_{\ell\le1} P(\xi=\ell)\alpha^{1-\ell}\right)\alpha^{-k-1}U \\ &= (\alpha)\alpha^{-k-1}U \quad \text{(by (2.2))} \\ &= \alpha^{-k}U = \widehat{V}(k). \end{aligned}$$

It follows from Lemma 2.1 that $\widehat{V}(k) \ge V(k)$, so $V(k) = \widehat{V}(k) > g(k)$. Since $e^{-\gamma n}V(X_n) = e^{-\gamma n}\widehat{V}(X_n)$ is a positive martingale, we have for all k and for any stopping rule τ,

$$V(k) = V(X_0) \ge E_k\left(e^{-\gamma\tau}V(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right) \ge E_k\left(e^{-\gamma\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right),$$

where the second inequality is strict if $P(\tau < \infty) > 0$. Hence, there exists no optimal stopping rule.

(iii) Since $u = -\infty$, we have $g(k-1)/g(k) \geq \alpha$ for all k, implying that $g(k) > 0$ for all k. By Lemma 2.1, it suffices to show that $g(k) \geq E[e^{-\gamma} g(k+\xi)]$. Note that

$$\frac{E[e^{-\gamma} g(k+\xi)]}{g(k)} = e^{-\gamma} \sum_{\ell \leq 1} P(\xi = \ell) \frac{g(k+\ell)}{g(k)}$$
$$\leq e^{-\gamma} \sum_{\ell \leq 1} P(\xi = \ell) \left(\frac{g(k-1)}{g(k)} \right)^{-\ell} = \frac{f(c')}{c'} + 1 \leq 1,$$

where the first inequality follows from (2.10), the second equality is by (2.3) with $c' := g(k-1)/g(k)$ and the last inequality is due to $f(c') \leq 0$, since $1 \geq c' = g(k-1)/g(k) \geq \alpha$ (*cf.* Remark 2.1). This proves that $E[e^{-\gamma} g(k+\xi)] \leq g(k)$ for all k. The proof is complete. □

Remark 2.2. *By setting $g(-\infty) = 0$, Theorem 2.1 can be readily extended to "defective" skip-free random walks with $P(\xi = -\infty) > 0$.*

Remark 2.3. *As pointed out by a referee, if the threshold u in (2.5) is not $+\infty$, the optimal stopping rule τ_u in Theorem 2.1 is a one-step-look-ahead rule for the associated problem for the ladder height process. Since g is assumed to be increasing and logconcave, the latter problem is a monotone stopping problem (cf. Chow, Robbins and Siegmund [4]). Theorem 2.1 shows that the original problem for the process $\{X_n\}$ and the associated problem for the ladder height process are equivalent.*

3 Optimal stopping rules for logconcave reward functions: the continuous case

Let $Y = \{Y_t\}_{t \geq 0}$ with $Y_0 = 0$ be a spectrally negative Lévy process defined on a probability space $(\Omega, \mathcal{F}, P)$. We consider the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ where $\mathcal{F}_t$ is the P-completed σ-field generated by $\{Y_s : 0 \leq s \leq t\}$, which satisfies the usual conditions. Assume that $P(Y_1 > 0) > 0$. (The reader is referred to [10] for a review of fluctuation theory of spectrally negative Lévy processes; see also [2, 15] for a complete discussion of Lévy processes.) In the absence of positive jumps, the Laplace exponent $\psi(\lambda)$ is well defined for all $\lambda \geq 0$, i.e.

$$E[e^{\lambda Y_t}] = e^{t\psi(\lambda)} \text{ for } \lambda \geq 0 \text{ and } t \geq 0. \tag{3.1}$$

Clearly $\psi(0) = 0$ and ψ is convex and tends to infinity as $\lambda \to \infty$. For $\gamma \geq 0$, let $\Phi(\gamma) = \sup\{\lambda \geq 0 : \psi(\lambda) = \gamma\}$, the largest (nonnegative) root of the equation $\psi(\lambda) = \gamma$, which is positive for $\gamma > 0$. For $\gamma = 0$, note that since $\psi(0) = 0, \psi'(0+) = E(Y_1)$ and ψ is convex, we have $\Phi(0) > 0$ if and only if $E(Y_1) < 0$.

With $x \in \mathbb{R}$, let $X_t = x + Y_t$ for $t \geq 0$, which is a Lévy process with initial state $X_0 = x$. For $a \in \mathbb{R}$, define $\tau_a = \inf\{t \geq 0 : X_t \geq a\}$ (a stopping rule of threshold form). Then in the absence of positive jumps, for $X_0 = x \leq a$, we have $X_{\tau_a} = a$ a.s. on $\{\tau_a < \infty\}$, and (*cf.* [10, Equation (3)])

$$E_x\left(e^{-\gamma\tau_a}\mathbf{1}_{\{\tau_a<\infty\}}\right) = e^{-\Phi(\gamma)(a-x)}, \tag{3.2}$$

where the subscript x in E_x refers to the initial state $X_0 = x$. It follows from (3.1) and the definition of Φ that

$$E_x[e^{\lambda(X_t-x)}] = e^{t\psi(\lambda)} \text{ for all } \lambda \geq 0, \text{ and } \psi(\Phi(\gamma)) = \gamma. \tag{3.3}$$

Consider a nonnegative reward function $g : \mathbb{R} \to [0, \infty)$ which is nonconstant, increasing (i.e. $g(x) \leq g(y)$ for $x \leq y$) and logconcave (i.e. $g(\theta x + (1-\theta)y) \geq (g(x))^\theta (g(y))^{1-\theta}$ for all x, y and $0 < \theta < 1$). Letting $\log 0 := -\infty$, the function $h(x) := \log g(x)$ is increasing and concave, so that the left-hand derivative $h'(x-)$ is well defined (possibly $+\infty$) at every x with $h(x) > -\infty$. Letting $h'(x-) := +\infty$ if $h(x) = -\infty$, we have that $h'(x-)$ is decreasing (and nonnegative) in $x \in \mathbb{R}$. Define

$$w = w(\gamma) = \inf\{x \in \mathbb{R} : h'(x-) \leq \Phi(\gamma)\}, \tag{3.4}$$

$$W = W(\gamma) = \sup\left\{e^{-\Phi(\gamma)x}g(x) : x \in \mathbb{R}\right\}, \tag{3.5}$$

$$V^*(x) = V^*_\gamma(x) = \sup_{\tau\in\mathcal{M}} E_x\left(e^{-\gamma\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right) \ (x \in \mathbb{R}), \tag{3.6}$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0, \infty]$ (with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$). The following result is the continuous-time counterpart of Theorem 2.1.

Theorem 3.1. *Let $\gamma \geq 0$, and assume $E(Y_1) < 0$ if $\gamma = 0$, so that $\Phi(\gamma) > 0$. Let $g : \mathbb{R} \to [0, \infty)$ be nonconstant, increasing, logconcave and right-continuous, and define $w = w(\gamma), W = W(\gamma)$ and $V^*(x) = V^*_\gamma(x)$ as in (3.4) – (3.6). Then the following statements hold.*

(i) *If $-\infty < w < \infty$, then the threshold-form stopping rule τ_w is optimal, and $V^*(x) = g(x)$ for $x \geq w$; and $V^*(x) = e^{-\Phi(\gamma)(w-x)}g(w)$ for $x < w$.*

(ii) *If $w = \infty$, then $V^*(x) = e^{\Phi(\gamma)x}W$ for all x. If, in addition, $W = \infty$, then there exist (randomized) stopping rules that have an infinite expected (discounted) reward; if $W < \infty$, then there is no optimal stopping rule.*

(iii) *If $w=-\infty$, then $V^*(x)=g(x)$ for all x and the optimal stopping rule is to stop immediately.*

To prove Theorem 3.1, we need the following standard result which is the continuous-time analogue of Lemma 2.1 and can be established easily by observing that $\{e^{-\gamma t}f(X_t)\}_{t\geq 0}$ is a supermartingale.

Lemma 3.1. *Let $f(x)$ and $g(x)$ be nonnegative measurable functions defined on $\mathbb{R}$ and $\gamma\geq 0$. If $f(x)\geq g(x)$ and $f(x)\geq E_x[e^{-\gamma t}f(X_t)]$ for all $x\in\mathbb{R}$ and $t>0$, then*

$$f(x)\geq \sup_{\tau\in\mathcal{M}} E_x\left(e^{-\gamma\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right),\ x\in\mathbb{R}.$$

Proof of Theorem 3.1. (i) Let $\widehat{V}(x)=E_x\left(e^{-\gamma\tau_w}g(X_{\tau_w})\mathbf{1}_{\{\tau_w<\infty\}}\right),\ x\in\mathbb{R}$. Then by (3.2), $\widehat{V}(x)=g(x)$ for $x\geq w$ and $\widehat{V}(x)=e^{-\Phi(\gamma)(w-x)}g(w)$ for $x<w$. We need to prove $V^*(x)=\widehat{V}(x)$ for all x. Since clearly $V^*(x)\geq\widehat{V}(x)$, it remains to prove $\widehat{V}(x)\geq V^*(x)$. By Lemma 3.1, it suffices to show

$$\widehat{V}(x)>g(x)\text{ for }x<w, \tag{3.7}$$

and

$$\widehat{V}(x)\geq E_x[e^{-\gamma t}\widehat{V}(X_t)]\ \text{ for }\ x\in\mathbb{R}\ \text{ and } t>0. \tag{3.8}$$

By the definition of w, we have $0\leq h'(y-)\leq\Phi(\gamma)$ for all $y>w$, which implies that $0\leq h(y)-h(x)\leq\Phi(\gamma)(y-x)$ for all $y>x>w$. Pick an arbitrary y with $y>w$ and $g(y)>0$. Then we have by the right-continuity of g that

$$h(y)-h(w)=\lim_{x\to w+}(h(y)-h(x))\leq\lim_{x\to w+}\Phi(\gamma)(y-x)=\Phi(\gamma)(y-w)<+\infty,$$

implying that $h(w)>-\infty$. So we have $g(w)>0$ and $\widehat{V}(x)>0$ for all x.

To prove (3.7), let $b(x)=\frac{g(x)}{\widehat{V}(x)}$ for all $x\in\mathbb{R}$. For $x<w$, $\log b(x)=h(x)+\Phi(\gamma)(w-x)-h(w)$, whose left-hand derivative equals $h'(x-)-\Phi(\gamma)>0$, implying that $b(x)$ is increasing in $-\infty<x<w$ and strictly increasing in $x_0<x<w$ where $x_0:=\inf\{y:g(y)>0\}\leq w$. Since $b(w-)=g(w-)/\widehat{V}(w)\leq g(w)/\widehat{V}(w)=1$, we have $b(x)<1$, i.e. $\widehat{V}(x)>g(x)$ for $x<w$. This prove (3.7).

To prove (3.8), let $a(y)=e^{-\Phi(\gamma)(w-y)}g(w)$ and $q(y)=\log\left(\frac{\widehat{V}(y)}{a(y)}\right)$ for all $y\in\mathbb{R}$. Since $q(y)=h(y)+\Phi(\gamma)(w-y)-h(w)$ for $y>w$, we have $q'(y-)=h'(y-)-\Phi(\gamma)\leq 0$ for $y>w$, implying that $q(y)$ is decreasing in $y>w$. Since $q(w)=0$ and $g(y)$ (and hence $q(y)$) is right-continuous at $y = w$, we have $q(y) \le 0$ for $y > w$, i.e. $\widehat{V}(y) \le a(y)$ for $y > w$. Since $\widehat{V}(y) = a(y)$ for $y \le w$, we have $\widehat{V}(y) \le a(y)$ for all $y \in \mathbb{R}$. Now for $X_0 = x \le w$,

$$\begin{aligned}
E_x[e^{-\gamma t}\widehat{V}(X_t)] &\le E_x[e^{-\gamma t}a(X_t)] \\
&= g(w)e^{-\gamma t}E_x[e^{-\Phi(\gamma)(w-X_t)}] \\
&= g(w)e^{-\Phi(\gamma)(w-x)-\gamma t}E_x[e^{\Phi(\gamma)(X_t-x)}] \\
&= g(w)e^{-\Phi(\gamma)(w-x)-\gamma t+\psi(\Phi(\gamma))t} \\
&= g(w)e^{-\Phi(\gamma)(w-x)} = \widehat{V}(x),
\end{aligned}$$

where the third and fourth equalities are by (3.3).

It remains to show that $\widehat{V}(x) \ge E_x[e^{-\gamma t}\widehat{V}(X_t)]$ for $X_0 = x > w$. Letting $\widehat{h}(y) = \log \widehat{V}(y)$ for all $y \in \mathbb{R}$, note that $\widehat{h}(y) = h(y)$ for $y \ge w$ and $\widehat{h}(y) = h(w) - \Phi(\gamma)(w - y)$ for $y < w$. Since $\widehat{h}'(y) = \Phi(\gamma)$ for $y < w$ and $h(y)$ (and hence $\widehat{h}(y)$) is concave in $y > w$ and $\widehat{h}'(y-) = h'(y-) \le \Phi(\gamma)$ for $y > w$, we have that $\widehat{h}(y)$ is increasing and concave on $\mathbb{R}$. It follows that

$$\widehat{h}(y) \le h(x) + \widehat{c}(y - x) \text{ for all } y \in \mathbb{R},$$

where $0 \le \widehat{c} := \widehat{h}'(x-) = h'(x-) \le \Phi(\gamma)$ (since $x > w$). So

$$\begin{aligned}
E_x[e^{-\gamma t}\widehat{V}(X_t)] &= e^{-\gamma t}E_x[e^{\widehat{h}(X_t)}] \\
&\le e^{-\gamma t}E_x[e^{h(x)+\widehat{c}(X_t-x)}] \\
&= e^{h(x)-\gamma t}E_x[e^{\widehat{c}(X_t-x)}] \\
&= e^{h(x)-\gamma t+\psi(\widehat{c})t} \quad \text{(by (3.3))} \\
&\le e^{h(x)-\gamma t+\psi(\Phi(\gamma))t} \\
&= e^{h(x)} = g(x) = \widehat{V}(x),
\end{aligned}$$

where the second inequality follows since $0 \le \widehat{c} \le \Phi(\gamma)$ and $\psi(z) \le \max\{\psi(0), \psi(\Phi(\gamma))\} = \psi(\Phi(\gamma))$ for all $0 \le z \le \Phi(\gamma)$ (by convexity of ψ). The proof of part (i) is complete.

(ii) Let $\widehat{V}(x) = e^{\Phi(\gamma)x}W$ for all x. We need to show $V^*(x) = \widehat{V}(x)$. Since $w = \infty$, the left-hand derivative of $h(x) - \Phi(\gamma)x$ is $h'(x-) - \Phi(\gamma) > 0$ for all x, so that $e^{-\Phi(\gamma)x}g(x) = e^{h(x)-\Phi(\gamma)x}$ is strictly increasing in $x \ge x_0 := \inf\{y : g(y) > 0\}$. Since g is nonconstant, we have $x_0 < \infty$ and $W = \sup\{e^{-\Phi(\gamma)y}g(y) : y \in \mathbb{R}\} = \lim_{y\to\infty} e^{-\Phi(\gamma)y}g(y) > e^{-\Phi(\gamma)x}g(x)$ for all $x \in \mathbb{R}$, implying that $\widehat{V}(x) > g(x)$ for all $x \in \mathbb{R}$. For $X_0 = x < y$, we have

$$\begin{aligned} V^*(x) &\geq E_x\left(e^{-\gamma\tau_y} g(X_{\tau_y}) \mathbf{1}_{\{\tau_y<\infty\}}\right) \\ &= g(y) E_x(e^{-\gamma\tau_y} \mathbf{1}_{\{\tau_y<\infty\}}) = e^{-\Phi(\gamma)(y-x)} g(y) \to e^{\Phi(\gamma)x} W = \widehat{V}(x) \text{ as } y \to \infty. \end{aligned}$$

So $V^*(x) \geq \widehat{V}(x)$ for all x. The remaining claims for part (ii) can be established by treating the cases $W = \infty$ and $W < \infty$ separately along the lines of the proof of Theorem 2.1(ii). The details are omitted.

(iii) By Lemma 3.1, it suffices to show that for $x \in \mathbb{R}$ and $t > 0$,

$$g(x) \geq E_x[e^{-\gamma t} g(X_t)], \tag{3.9}$$

which can be established along the lines of the proof of (3.8) for $X_0 = x > w$ (*cf.* the last part of the proof of (i)). Briefly, since $w = -\infty$, we can argue that $g(x) > 0$ for all $x \in \mathbb{R}$. Since $h(x) = \log g(x)$ is increasing and concave on $\mathbb{R}$,

$$h(y) \leq h(x) + c(y - x) \text{ for all } x \text{ and } y, \tag{3.10}$$

where $0 \leq c := h'(x-) \leq \Phi(\gamma)$. Then (3.9) follows from (3.10). □

Remark 3.1. *If a nonnegative function $g : \mathbb{R} \to [0, \infty)$ is increasing and logconcave, it is easily shown that $g(x)$ is continuous everywhere except possibly at $x_0 := \inf\{x \in \mathbb{R} : g(x) > 0\}$. If $x_0 > -\infty$, then Theorem 3.1 requires that $g(x)$ be right-continuous at $x = x_0$, i.e. $g(x_0) = g(x_0+)$. This right-continuity condition cannot be removed as the following example shows. For $0 \leq c \leq 1$, consider $g_c : \mathbb{R} \to [0, \infty)$ defined by*

$$g_c(x) = \begin{cases} 0, & \text{if } x < 0; \\ c, & \text{if } x = 0; \\ 1, & \text{if } x > 0; \end{cases}$$

which is increasing and logconcave. It is readily seen that the value function $V^*(x) = V^*_\gamma(x) = \min\{e^{\Phi(\gamma)x}, 1\}$ for $x \in \mathbb{R}$, which is independent of $c \in [0, 1]$. For $c = 1$, g_c is right-continuous and the optimal stopping rule is τ_0. But for $0 \leq c < 1$, g_c is not right-continuous and no optimal stopping rule exists. On the other hand, for a nonnegative, increasing and logconcave reward function g which is not right-continuous at $x = x_0$ (i.e. $g(x_0-) = 0 \leq g(x_0) < g(x_0+)$), let $\widetilde{g}(x) := g(x)$ for $x \neq x_0$; and $\widetilde{g}(x_0) := g(x_0+)$, which is *increasing, logconcave and right-continuous. For the reward function* $\widetilde{g}$*, suppose the optimal threshold value* w *defined in* (3.4) *(with* $h'(x-)$ *replaced by the left-hand derivative of* $\log \widetilde{g}(x)$*) is such that* $x_0 < w < \infty$*. Then the stopping rule* τ_w *is optimal for the reward function* g *since* τ_w *is optimal for* $\widetilde{g}$ *and since* $g(x) = \widetilde{g}(x)$ *for* $x \neq x_0$*,* $g(x_0) < \widetilde{g}(x_0)$ *and* $x_0 < w$*. Moreover, the two reward functions* g *and* $\widetilde{g}$ *yield the same value function. As an example, consider a Brownian motion with drift parameter* $-a$ *(*$a > 0$*) and* $\gamma = 0$ *(without discounting). Then we have* $\psi(\lambda) = \lambda^2/2 - a\lambda$ *and* $\Phi(0) = 2a$*. For* $0 \leq c \leq 1$*, let* $g_c^* : \mathbb{R} \to [0, \infty)$ *be defined by*

$$g_c^*(x) = \begin{cases} 0, & \text{if } x < 0; \\ c, & \text{if } x = 0; \\ e^{\sqrt{x}}, & \text{if } x > 0; \end{cases} \tag{3.11}$$

which is increasing and logconcave for all $0 \leq c \leq 1$ *and is right-continuous at* $x_0 = 0$ *if and only if* $c = 1$*. For the reward function* g_1^**, the optimal threshold value defined in* (3.4) *is* $w = 1/(16a^2) > 0 = x_0$*. It follows that* τ_w *is optimal for the reward function* g_c^* *for all* $0 \leq c \leq 1$*.*

Remark 3.2. *Theorem 2.1 is concerned with the discrete-time discrete-state case while Theorem 3.1 deals with the continuous-time continuous-state case. We now consider the continuous-time discrete-state case involving a compound Poisson process*

$$X_t = k + \sum_{i=1}^{N_t} \xi_i \ (t \geq 0), \ X_0 = k \in \mathbb{Z},$$

where $\xi, \xi_1, \xi_2, \dots$ *is a sequence of independent and identically distributed integer-valued random variables with* $P(\xi > 1) = 0$ *and* $P(\xi = 1) > 0$*, and* $\{N_t\}_{t \geq 0}$ *is a Poisson process with constant rate* $\mu > 0$ *which is independent of the* ξ_i'*s.*

Let $\phi(\lambda) = E[e^{\lambda \xi}]$*,* $\lambda \geq 0$*. Clearly,* $\phi(0) = 1$ *and* ϕ *is convex and tends to infinity as* $\lambda \to \infty$*. For* $\gamma \geq 0$ *and* $\mu > 0$*, let* $\beta = \beta(\gamma, \mu) = \sup\left\{\lambda \geq 0 : \phi(\lambda) = 1 + \frac{\gamma}{\mu}\right\}$*, which is positive for* $\gamma > 0$*. For* $\gamma = 0$*, note that since* $\phi(0) = 1$*,* $\phi'(0+) = E(\xi)$ *and* ϕ *is convex, we have* $\beta > 0$ *if and only if* $E(\xi) < 0$*.*

For a (nonnegative) reward function $g : \mathbb{Z} \to [0, \infty)$ *which is nonconstant, increasing and* *logconcave, define*

$$z = z(\gamma,\mu) = \inf\left\{k \in \mathbb{Z} : \frac{g(k)}{g(k+1)} \geq e^{-\beta}\right\}, \tag{3.12}$$

$$Z = Z(\gamma,\mu) = \sup\left\{e^{-\beta\ell} g(\ell) : \ell \in \mathbb{Z}\right\}, \tag{3.13}$$

$$V^{**}(k) = V^{**}_{\gamma,\mu}(k) = \sup_{\tau\in\mathcal{M}} E_k\left(e^{-\gamma\tau} g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right), \tag{3.14}$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0,\infty]$. Then we have the following result which can be established along the lines of the proof of Theorem 2.1.

Theorem 3.2. *Let $\gamma \geq 0$, and assume $E(\xi) < 0$ if $\gamma = 0$, so that $\beta > 0$. Let $g : \mathbb{Z} \to [0,\infty)$ be nonconstant, increasing and logconcave, and define $z = z(\gamma,\mu)$, $Z = Z(\gamma,\mu)$ and $V^{**}(k) = V^{**}_{\gamma,\mu}(k)$ as in (3.12) – (3.14). Then the following statements hold.*

(i) *If $-\infty < z < \infty$, then the threshold-form stopping rule τ_z is optimal, and $V^{**}(k) = g(k)$ for $k \geq z$; and $V^{**}(k) = e^{-\beta(z-k)}g(z)$ for $k < z$.*

(ii) *If $z = \infty$, then $V^{**}(k) = e^{\beta k}Z$ for all k. If, in addition, $Z = \infty$, then there exist (randomized) stopping rules that have an infinite expected (discounted) reward; if $Z < \infty$, then there is no optimal stopping rule.*

(iii) *If $z = -\infty$, then $V^{**}(k) = g(k)$ for all k and the optimal stopping rule is to stop immediately.*

4 Necessity of logconcavity and monotonicity: the discrete case

Theorem 2.1 shows that for a nonnegative, increasing and logconcave reward function g, the optimal stopping rule is of threshold form under a general skip-free random walk model. In this section, we prove a converse of Theorem 2.1 by restricting attention to Bernoulli random walks without discounting (i.e. $\gamma = 0$). Specifically, let $\{X_n\}_{n\geq 0}$ be a Bernoulli random walk with parameter p (denoted BRW(p))

$$X_{n+1} = X_n + \xi_{n+1},\ n = 0, 1, \ldots;\ X_0 = k \in \mathbb{Z},$$

where $\xi, \xi_1, \xi_2, \ldots$ are independent and identically distributed with $P(\xi = 1) = p$, $P(\xi = -1) = q = 1 - p$ ($0 < p < \frac{1}{2}$). For $u \in \mathbb{Z} \cup \{-\infty\}$, let $\tau_u = \inf\{n \geq 0 : X_n \geq u\}$. (Note that

$\tau_{-\infty} = 0$.) For $u \in \mathbb{Z} \cup \{-\infty\}$ and $g : \mathbb{Z} \to [0, \infty)$, it is well known that

$$E_{k,p}\left[g(X_{\tau_u})\mathbf{1}_{\{\tau_u<\infty\}}\right] = \begin{cases} g(u)(p/q)^{u-k}, & \text{if } k < u; \\ g(k), & \text{if } k \geq u, \end{cases} \tag{4.1}$$

where the subscripts k and p in $E_{k,p}$ refer to the initial state $X_0 = k$ and the parameter p of Bernoulli random walk BRW(p).

Definition 4.1. *Let $g : \mathbb{Z} \to [0, \infty)$ be a reward function and*

$$V(k, p) = \sup_{\tau \in \mathcal{M}} E_{k,p}[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}],$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0, \infty]$. We say that g is of **threshold type** *with respect to Bernoulli random walks if for each $p \in (0, \frac{1}{2})$ there is an $m(p) \in \mathbb{Z} \cup \{-\infty\}$ such that the stopping rule $\tau_{m(p)}$ is optimal under the BRW(p) model, i.e. for all $p \in (0, \frac{1}{2})$ and $k \in \mathbb{Z}$,*

$$V(k, p) = E_{k,p}[g(X_{\tau_{m(p)}})\mathbf{1}_{\{\tau_{m(p)}<\infty\}}]. \tag{4.2}$$

We now present the main result in this section.

Theorem 4.1. *If $g : \mathbb{Z} \to [0, \infty)$ is of threshold type with respect to Bernoulli random walks, then g is increasing and logconcave.*

The key to the proof of Theorem 4.1 is the following lemma.

Lemma 4.1. *Suppose $g : \mathbb{Z} \to [0, \infty)$ is not identically 0 and is of threshold type with respect to Bernoulli random walks (i.e. g satisfies (4.2)). Then the following properties hold.*

(i) *For all $k \in \mathbb{Z}$ and $p \in (0, \frac{1}{2})$, $V(k, p) > 0$ and*

$$g(k) \leq V(k, p) = \begin{cases} g(m(p))(p/q)^{m(p)-k}, & \text{if } k < m(p); \\ g(k), & \text{if } k \geq m(p), \end{cases}$$

(ii) *$m(p)$ is increasing in $p \in (0, \frac{1}{2})$, i.e. $m(p_1) \leq m(p_2)$ for $0 < p_1 < p_2 < \frac{1}{2}$,*

(iii) *$g(k) = 0$ for all $k < m_0 := \inf\left\{m(p) : 0 < p < \frac{1}{2}\right\}$.*

Proof. (i) It is clear that $V(k,p) \geq E_{k,p}[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}]$ for any stopping rule τ. Since g is of threshold type with respect to Bernouli random walks, we have by (4.1) and (4.2) that

$$g(k) \leq V(k,p) = \begin{cases} g(m(p))(p/q)^{m(p)-k}, & \text{if } k < m(p); \\ g(k), & \text{if } k \geq m(p). \end{cases}$$

Since g is not identically 0, $g(i_0) > 0$ for some $i_0 \in \mathbb{Z}$. Consider the stopping rule $\tau = \inf\{n \geq 0 : X_n = i_0\}$ (which is different from τ_{i_0} if $X_0 = k > i_0$). Then

$$\begin{aligned} V(k,p) &\geq E_{k,p}\left[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \\ &= g(i_0)E_{k,p}\left[\mathbf{1}_{\{\tau<\infty\}}\right] \\ &= g(i_0)\min\left\{(p/q)^{i_0-k}, 1\right\} > 0 \end{aligned}$$

for all $k \in \mathbb{Z}$ and $p \in (0, \frac{1}{2})$. This proves (i).

(ii) Suppose $m(p_1) > m(p_2)$ for some $0 < p_1 < p_2 < \frac{1}{2}$. Letting $m_1 := m(p_1)$, $q_1 := 1-p_1$, $q_2 := 1 - p_2$, we have by (4.1) and part (i)

$$\begin{aligned} g(m_1)\left(\frac{p_2}{q_2}\right) &= E_{m_1-1,p_2}\left[g(X_{\tau_{m_1}})\mathbf{1}_{\{\tau_{m_1}<\infty\}}\right] \\ &\leq V(m_1-1,p_2) = g(m_1-1) \text{ (since } m_1 - 1 = m(p_1) - 1 \geq m(p_2)) \\ &\leq V(m_1-1,p_1) = g(m_1)\left(\frac{p_1}{q_1}\right), \end{aligned}$$

which together with $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ implies that $g(m_1) = 0$ and $V(m_1-1,p_1) = 0$, contradicting (i). This proves (ii).

(iii) It suffices to consider $m_0 > -\infty$. Then, by the definition of m_0 and (ii), there exists $p_0 \in (0, \frac{1}{2})$ such that $m(p) = m_0$ for all $0 < p \leq p_0$. For $0 < p \leq p_0$ and any $k < m_0(= m(p))$, we have by (i)

$$g(k) \leq V(k,p) = g(m_0)\left(\frac{p}{q}\right)^{m_0-k} \to 0 \text{ as } p \to 0,$$

proving (iii). □

Proof of Theorem 4.1. If g is identically zero, then we are done. Suppose now that g is not identically 0. We shall first show that g is increasing. Suppose to the contrary that $g(\ell) > g(\ell+1)$ (implying $g(\ell) > 0$) for some $\ell \in \mathbb{Z}$. By Lemma 4.1(iii), we have $\ell \geq m_0$ since $g(\ell) > 0$. Thus in either of the two cases $m_0 = -\infty$ and $m_0 > -\infty$, there exists $p \in (0, \frac{1}{2})$

such that $m(p) \le \ell$. Letting $\tau = \inf\{n \ge 0 : X_n = \ell\}$ and noting that $E_{\ell+1,p}[\mathbf{1}_{\{\tau<\infty\}}] = 1$, we have by Lemma 4.1(i)

$$g(\ell+1) = V(\ell+1,p) \ge E_{\ell+1,p}[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}] = g(\ell),$$

a contradiction. This proves that g is increasing.

It remains to show that g is logconcave. Suppose to the contrary that $(g(\ell+1))^2 < g(\ell)g(\ell+2)$ for some $\ell \in \mathbb{Z}$. Then $g(\ell) > 0$ and hence $g(k) > 0$ for all $k \ge \ell$ since g is increasing. Choose $p \in (0, \frac{1}{2})$ such that

$$\frac{g(\ell+1)}{g(\ell+2)} < \frac{p}{q} < \frac{g(\ell)}{g(\ell+1)}. \tag{4.3}$$

For $k \ge m(p)$, we have by Lemma 4.1(i) that $g(k+1) = V(k+1,p) > 0$ and

$$g(k) = V(k,p) \ge E_{k,p}\left[g(X_{\tau_{k+1}})\mathbf{1}_{\{\tau_{k+1}<\infty\}}\right] = g(k+1)\left(\frac{p}{q}\right).$$

So

$$\frac{g(k)}{g(k+1)} \ge \frac{p}{q} \text{ for all } k \ge m(p),$$

implying by (4.3) that $\ell+1 < m(p)$. Let $\ell_1, \ell_2 \in \{\ell+1, \ell+2, \dots, m(p)\}$ be such that

$$\left(\frac{g(\ell)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}} = \min\left\{\left(\frac{g(\ell)}{g(k)}\right)^{\frac{1}{k-\ell}} : k = \ell+1, \ell+2, \dots, m(p)\right\}, \tag{4.4}$$

$$\frac{g(\ell_2)}{g(\ell_2+1)} = \min\left\{\frac{g(k)}{g(k+1)} : k = \ell+1, \ell+2, \dots, m(p)\right\}. \tag{4.5}$$

By (4.3)–(4.5),

$$\frac{p}{q} > \frac{g(\ell+1)}{g(\ell+2)} \ge \frac{g(\ell_2)}{g(\ell_2+1)},$$

$$\begin{aligned}
\left(\frac{g(\ell)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}} &= \left(\frac{g(\ell)}{g(\ell+1)}\frac{g(\ell+1)}{g(\ell+2)}\cdots\frac{g(\ell_1-1)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}} \\
&> \left(\frac{g(\ell+1)}{g(\ell+2)}\left(\frac{g(\ell_2)}{g(\ell_2+1)}\right)^{\ell_1-\ell-1}\right)^{\frac{1}{\ell_1-\ell}} \\
&\ge \left(\left(\frac{g(\ell_2)}{g(\ell_2+1)}\right)^{\ell_1-\ell}\right)^{\frac{1}{\ell_1-\ell}} = \frac{g(\ell_2)}{g(\ell_2+1)},
\end{aligned}$$

from which follows

$$\min\left\{\frac{p}{q}, \left(\frac{g(\ell)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}}\right\} > \frac{g(\ell_2)}{g(\ell_2+1)}.$$

Choose $p' \in (0,p)$ such that

$$\min\left\{\frac{p}{q}, \left(\frac{g(\ell)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}}\right\} > \frac{p'}{q'} > \frac{g(\ell_2)}{g(\ell_2+1)}, \tag{4.6}$$

where $q' = 1-p'$. For $k \in \{\ell+1, \ell+2, \ldots, m(p)\}$, we have by (4.4) and (4.6)

$$\left(\frac{g(\ell)}{g(k)}\right)^{\frac{1}{k-\ell}} \geq \left(\frac{g(\ell)}{g(\ell_1)}\right)^{\frac{1}{\ell_1-\ell}} > \frac{p'}{q'},$$

so

$$V(\ell,p') \geq g(\ell) > g(k)\left(\frac{p'}{q'}\right)^{k-\ell},$$

implying by Lemma 4.1(i) that $m(p') \neq k$ for all $k \in \{\ell+1, \ell+2, \ldots, m(p)\}$. Since $m(p') \leq m(p)$ by Lemma 4.1(ii), we have $m(p') \leq \ell < \ell_2$, which together with (4.1), Lemma 4.1(i) and (4.6) implies that

$$E_{\ell_2,p'}[g(X_{\tau_{\ell_2+1}})\mathbf{1}_{\{\tau_{\ell_2+1}<\infty\}}] = g(\ell_2+1)\left(\frac{p'}{q'}\right) > g(\ell_2) = V(\ell_2,p'),$$

a contradiction. This proves that g is logconcave and completes the proof. □

5 Necessity of logconcavity and monotonicity: the continuous case

We have shown in Theorem 3.1 that for a nonnegative, increasing, logconcave and right-continuous reward function g, the optimal stopping rule is of threshold form with respect to a general spectrally negative Lévy process. In this section, we present a converse of Theorem 3.1 by restricting attention to Brownian motions without discounting. Specifically, let $\{X_t\}_{t\geq 0}$ be a Brownian motion with drift parameter $-a$ (denoted BM(a)), where $0 < a < \infty$, i.e.

$$X_t = x - at + B_t \text{ for } t \geq 0;\ X_0 = x \in \mathbb{R},$$

where x is the initial state and B_t is a standard Brownian motion. For a measurable function $g : \mathbb{R} \to [0,\infty)$ and $u \in \mathbb{R} \cup \{-\infty\}$, letting $\tau_u = \inf\{t \geq 0 : X_t \geq u\}$, we have the following standard result

$$E_{x,a}[g(X_{\tau_u})\mathbf{1}_{\{\tau_u<\infty\}}] = \begin{cases} g(u)e^{-2a(u-x)}, & \text{if } x < u; \\ g(x), & \text{if } x \geq u; \end{cases} \tag{5.1}$$

where the subscripts x and a in $E_{x,a}$ refer to the initial state $X_0 = x$ and the drift parameter $-a$ of Brownian motion BM(a).

Definition 5.1. *Let $g : \mathbb{R} \to [0, \infty)$ be a measurable reward function and*

$$V(x, a) = \sup_{\tau \in \mathcal{M}} E_{x,a} \left[g(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right],$$

where $\mathcal{M}$ is the class of all stopping rules τ with values in $[0, \infty]$. We say that g is of **threshold type** *with respect to Brownian motions if for each $a \in (0, \infty)$ there is a $u(a) \in \mathbb{R} \cup \{-\infty\}$ such that the stopping rule $\tau_{u(a)}$ is optimal under BM(a), i.e. for all $a \in (0, \infty)$ and $x \in \mathbb{R}$,*

$$V(x, a) = E_{x,a} \left[g(X_{\tau_{u(a)}}) \mathbf{1}_{\left\{\tau_{u(a)} < \infty\right\}} \right]. \tag{5.2}$$

The following result is the counterpart of Theorem 4.1 in the continuous case.

Theorem 5.1. *If a measurable reward function $g : \mathbb{R} \to [0, \infty)$ is of threshold type with respect to Brownian motions, then g is increasing and logconcave.*

We first establish the following lemma which is the key to the proof of Theorem 5.1.

Lemma 5.1. *Suppose a measurable reward function $g : \mathbb{R} \to [0, \infty)$ is not identically 0 and is of threshold type with respect to Brownian motions (i.e. g satisfies (5.2)). Then the following properties hold.*

(i) *For all $x \in \mathbb{R}$ and $a \in (0, \infty)$, $V(x, a) > 0$ and*

$$g(x) \leq V(x, a) = \begin{cases} g(u(a)) e^{-2a(u(a)-x)}, & \text{if } x < u(a); \\ g(x), & \text{if } x \geq u(a), \end{cases}$$

(ii) *$u(a)$ is decreasing in $a \in (0, \infty)$, i.e. $u(a_1) \geq u(a_2)$ for $0 < a_1 < a_2 < \infty$,*

(iii) *$g(x) = 0$ for $x < u_\infty := \inf \{u(a) : 0 < a < \infty\}$.*

Proof. (i) It is clear that $V(x, a) \geq E_{x,a}[g(X_\tau) \mathbf{1}_{\{\tau < \infty\}}]$ for all stopping rules $\tau \in \mathcal{M}$. Since g is of threshold type with respect to Brownian motions, we have by (5.1) and (5.2) that

$$g(x) \leq V(x, a) = \begin{cases} g(u(a)) e^{-2a(u(a)-x)}, & \text{if } x < u(a); \\ g(x), & \text{if } x \geq u(a). \end{cases}$$

Since g is not identically 0, $g(x') > 0$ for some $x' \in \mathbb{R}$. Consider the stopping rule $\tau = \inf\{t \geq 0 : X_t = x'\}$. Then

$$\begin{aligned} V(x,a) &\geq E_{x,a}[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}] \\ &= g(x')E_{x,a}[\mathbf{1}_{\{\tau<\infty\}}] \\ &= g(x')\min\{e^{-2a(x'-x)}, 1\} > 0 \end{aligned}$$

for all $x \in \mathbb{R}$ and $a \in (0,\infty)$. This proves (i).

(ii) Suppose $u(a_1) < u(a_2)$ for some $0 < a_1 < a_2 < \infty$. Letting $u_2 := u(a_2)$ and $\delta \in (0, u(a_2) - u(a_1))$ (so that $u(a_1) < u(a_2) - \delta = u_2 - \delta < u_2$), we have by (5.1) and part (i)

$$\begin{aligned} e^{-2a_1\delta}g(u_2) &= E_{u_2-\delta,a_1}\left[g(X_{\tau_{u_2}})\mathbf{1}_{\{\tau_{u_2}<\infty\}}\right] \\ &\leq V(u_2-\delta,a_1) = g(u_2-\delta) \\ &\leq V(u_2-\delta,a_2) = e^{-2a_2\delta}g(u_2), \end{aligned}$$

which together with $0 < a_1 < a_2$ implies that $g(u_2) = 0$ and $V(u_2 - \delta, a_2) = 0$, contradicting (i). This proves (ii)

(iii) It suffices to consider $u_\infty > -\infty$. Pick an (arbitrary) $a_3 \in (0,\infty)$ and let $u_3 := u(a_3) \geq u_\infty$. We have by (i)

$$g(y) \leq V(y,a_3) = e^{-2a_3(u_3-y)}g(u_3) \text{ for all } y \leq u_3. \tag{5.3}$$

For any $x < u_\infty$, we have by (i), (ii) and (5.3) that for all $a > a_3$

$$0 \leq g(x) \leq V(x,a) = e^{-2a(u(a)-x)}g(u(a)) \leq e^{-2a(u(a)-x)-2a_3(u_3-u(a))}g(u_3),$$

which converges to 0 as $a \to \infty$. This proves (iii). □

Proof of Theorem 5.1. If g is identically zero, then we are done. Suppose now that g is not identically 0. We shall first show that g is increasing. Suppose to the contrary that $g(r) > g(s)$ (implying $g(r) > 0$) for some $r < s$. By Lemma 5.1(iii), we have $u_\infty \leq r < s$, so that there exists $a \in (0,\infty)$ such that $u(a) < s$. Letting $\tau = \inf\{t \geq 0 : X_t = r\}$ and noting that $E_{s,a}\left[\mathbf{1}_{\{\tau<\infty\}}\right] = 1$, we have by Lemma 5.1(i)

$$g(s) = V(s,a) \geq E_{s,a}\left[g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = g(r),$$

a contradiction. This proves that g is increasing.

It remains to show that g is logconcave. Suppose to the contrary that there exist $r < c < s$ such that

$$g(c) < g(r)^{\frac{s-c}{s-r}} g(s)^{\frac{c-r}{s-r}}. \tag{5.4}$$

Then $g(r) > 0$ and hence $g(x) > 0$ for all $x \geq r$ since g is increasing. By (5.4),

$$0 < \left(\frac{g(c)}{g(s)}\right)^{\frac{1}{s-c}} < \left(\frac{g(r)}{g(s)}\right)^{\frac{1}{s-r}} < \left(\frac{g(r)}{g(c)}\right)^{\frac{1}{c-r}} \leq 1. \tag{5.5}$$

We claim that g is continuous on (r, ∞). Since g is increasing, it suffices to show that $g(x-) \geq g(x+)$ for all $x > r$. Since Lemma 5.1(iii) together with $g(r) > 0$ implies that $u_\infty \leq r < x$, there exists $a \in (0, \infty)$ such that $u(a) < x$. By Lemma 5.1(i), we have for $0 < \varepsilon < x - u(a)$ (implying $x - \varepsilon > u(a)$),

$$\begin{aligned} g(x-\varepsilon) = V(x-\varepsilon, a) &\geq E_{x-\varepsilon,a}\left[g(X_{\tau_{x+\varepsilon}})\mathbf{1}_{\{\tau_{x+\varepsilon}<\infty\}}\right] \\ &= e^{-2a(2\varepsilon)} g(x+\varepsilon), \end{aligned}$$

which by letting $\varepsilon \to 0$ yields $g(x-) \geq g(x+)$. Thus, g is continuous on (r, ∞).

In view of (5.5), let $a' \in (0, \infty)$ be such that

$$\left(\frac{g(c)}{g(s)}\right)^{\frac{1}{s-c}} < e^{-2a'} < \left(\frac{g(r)}{g(s)}\right)^{\frac{1}{s-r}} < \left(\frac{g(r)}{g(c)}\right)^{\frac{1}{c-r}}. \tag{5.6}$$

For $y > x \geq u(a')$, we have by Lemma 5.1(i) that $g(y) = V(y, a') > 0$ and

$$g(x) = V(x, a') \geq E_{x,a'}\left[g(X_{\tau_y})\mathbf{1}_{\{\tau_y<\infty\}}\right] = e^{-2a'(y-x)} g(y) > 0.$$

So,

$$\left(\frac{g(x)}{g(y)}\right)^{\frac{1}{y-x}} \geq e^{-2a'} \text{ for all } y > x \geq u(a'),$$

implying by (5.6) that $c < u(a')$. Let

$$L = \inf\left\{\left(\frac{g(c)}{g(x)}\right)^{\frac{1}{x-c}} : c < x \leq \max\{s, u(a')\}\right\}. \tag{5.7}$$

Since $\left(\frac{g(r)}{g(x)}\right)^{\frac{1}{x-r}}$ is continuous in $x \in [c, \max\{s, u(a')\}]$, there is an $x_1 \in [c, \max\{s, u(a')\}]$ such that

$$\left(\frac{g(r)}{g(x_1)}\right)^{\frac{1}{x_1-r}} = \min\left\{\left(\frac{g(r)}{g(x)}\right)^{\frac{1}{x-r}} : c \leq x \leq \max\{s, u(a')\}\right\}. \tag{5.8}$$

By (5.6) and (5.7),

$$e^{-2a'} > \left(\frac{g(c)}{g(s)}\right)^{\frac{1}{s-c}} \geq L. \tag{5.9}$$

Note that $c \leq x_1 \leq \max\{s, u(a')\}$. By (5.6)–(5.8), if $x_1 = c$,

$$\left(\frac{g(r)}{g(x_1)}\right)^{\frac{1}{x_1-r}} = \left(\frac{g(r)}{g(c)}\right)^{\frac{1}{c-r}} > \left(\frac{g(c)}{g(s)}\right)^{\frac{1}{s-c}} \geq L; \tag{5.10}$$

if $x_1 > c$,

$$\begin{aligned}\left(\frac{g(r)}{g(x_1)}\right)^{\frac{1}{x_1-r}} &= \left[\left(\frac{g(c)}{g(x_1)}\right)^{\frac{1}{x_1-c}}\right]^{\frac{x_1-c}{x_1-r}} \left[\left(\frac{g(r)}{g(c)}\right)^{\frac{1}{c-r}}\right]^{\frac{c-r}{x_1-r}} \\ &> L^{\frac{x_1-c}{x_1-r}} \left[\left(\frac{g(c)}{g(s)}\right)^{\frac{1}{s-c}}\right]^{\frac{c-r}{x_1-r}} \\ &\geq L^{\frac{x_1-c}{x_1-r}} L^{\frac{c-r}{x_1-r}} = L. \end{aligned} \tag{5.11}$$

In view of (5.9)–(5.11), let $a'' \in (a', \infty)$ be such that

$$\min\left\{e^{-2a'}, \left(\frac{g(r)}{g(x_1)}\right)^{\frac{1}{x_1-r}}\right\} > e^{-2a''} > L. \tag{5.12}$$

For $y \in [c, u(a')]$, we have by (5.8) and (5.12)

$$\left(\frac{g(r)}{g(y)}\right)^{\frac{1}{y-r}} \geq \left(\frac{g(r)}{g(x_1)}\right)^{\frac{1}{x_1-r}} > e^{-2a''},$$

so $V(r, a'') \geq g(r) > e^{-2a''(y-r)} g(y)$, implying by Lemma 5.1(i) that $u(a'') \neq y$ for all $y \in [c, u(a')]$. Since $u(a'') \leq u(a')$ by Lemma 5.1(ii), we have $u(a'') < c$. It follows from (5.7) and (5.12) that $e^{-2a''} > \left(\frac{g(c)}{g(y_0)}\right)^{\frac{1}{y_0-c}}$ for some $y_0 \in (c, \max\{s, u(a')\}]$, which together with $u(a'') < c$ implies that

$$V(c, a'') = g(c) < e^{-2a''(y_0-c)} g(y_0) = E_{c,a''}\left[g(X_{\tau_{y_0}}) \mathbf{1}_{\{\tau_{y_0} < \infty\}}\right],$$

a contradiction. This proves that g is logconcave and completes the proof. □

Remark 5.1. *Under the assumptions on g in Theorem 5.1, g has been shown to be increasing and logconcave, so that it is continuous everywhere except possibly at $x_0 := \inf\{x : g(x) > 0\}$ (cf. Remark 3.1). However, g need not be right-continuous under the assumptions of Theorem 5.1, as the function g_c^* given in* (3.11) *(with $0 \leq c < 1$) shows.*

6 Concluding remarks

We have explored the close connection between increasing logconcave reward functions and optimal stopping rules of threshold form, which yields a partial characterization of the threshold structure of optimal stopping rules. In the discrete (continuous, *resp.*) case, we established that if a nonnegaive measurable reward function defined on $\mathbb{Z}$ ($\mathbb{R}$, *resp.*) is increasing and logconcave (and right-continuous for the continuous case), then the optimal stopping rule is of threshold form provided the underlying process is a skip-free random walk (a spectrally negative Lévy process, *resp.*). As these results only cover optimal stopping problems without overshoot, it would be of great interest to find, for problems with overshoot, (general) conditions on the reward function (in addition to logconcavity and monotonicity) under which the optimal stopping rule is of threshold form, so as to provide a more complete characterization of the threshold structure of optimal stopping rules.

In the case without discounting ($\gamma = 0$), we also established the necessity of logconcavity and monotonicity of a reward function in order for the optimal stopping rule to be of threshold form in the discrete (continuous, *resp.*) case when the underlying process belongs to the class of Bernoulli random walks (Brownian motions, *resp.*) with a downward drift. For the remainder of this section, we briefly address the issue of necessity of logconcavity and monotonicity when the discount rate γ is positive. With $\gamma > 0$ fixed, in the discrete case, we consider the class of Bernoulli random walks BRW(p) for all $0 < p < 1$. Let g be a positive reward function defined on $\mathbb{Z}$ with $g(k) \geq e^{-\gamma} \max\{g(k-1), g(k+1)\}$ for all $k \in \mathbb{Z}$. (Such a function g can be neither increasing nor logconcave.) Since $g(k) \geq E[e^{-\gamma} g(k+\xi)]$ where $P(\xi = 1) = p = 1 - q = 1 - P(\xi = -1)$, we have by Lemma 2.1

$$g(k) \geq E_{k,p}(e^{-\gamma\tau} g(X_\tau) \mathbf{1}_{\{\tau<\infty\}}), \quad \text{for all } k \in \mathbb{Z} \text{ and } \tau \in \mathcal{M},$$

implying that the stopping rule $\tau_{-\infty}$ (i.e. to stop immediately) is optimal under BRW(p) for all $0 < p < 1$. In particular, for a positive *increasing* reward function g with $g(k) \geq e^{-\gamma} g(k+1)$ for all k (which need not be logconcave), the stopping rule $\tau_{-\infty}$ is optimal under BRW(p) for all $0 < p < 1$. More generally, let g be a positive increasing reward function such that for some $k_0 \in \mathbb{Z}$, g is logconcave in $k \leq k_0$ (i.e. $g(k-2)/g(k-1) \leq g(k-1)/g(k)$ for all $k \leq k_0$) and $g(k) \geq e^{-\gamma} g(k+1)$ for all $k \geq k_0$. Then the optimal stopping rule is of threshold form under BRW(p) for all $0 < p < 1$. More precisely, for $0 < p < 1$, define

$$\alpha(p) = (e^{\gamma} - \sqrt{e^{2\gamma} - 4pq})/(2q),$$

which satisfies (2.2). Let

$$m(p) = \inf\{k \in \mathbb{Z} : g(k)/g(k+1) \geq \alpha(p)\}.$$

Note that $m(p) \leq k_0$ since $\alpha(p) \leq e^{-\gamma}$. Then it can be shown (along the lines of the proof of Theorem 2.1) that $\tau_{m(p)}$ is optimal under BRW(p) for all $0 < p < 1$. (This can also be argued by considering the associated problem for the ladder height process, which as a consequence of the conditions on g, is a monotone stopping problem; *cf.* Remark 2.3.)

In the continuous case, we consider the class of Brownian motions BM(a) for all $a \in \mathbb{R}$ where a is the (negative) drift parameter, i.e. $X_t = x - at + B_t$ for $t \geq 0$. For a measurable function $g : \mathbb{R} \to [0, \infty)$ and $u \in \mathbb{R} \cup \{-\infty\}$, we have the following standard result

$$E_{x,a}[e^{-\gamma\tau_u} g(X_{\tau_u})\mathbf{1}_{\{\tau_u<\infty\}}] = \begin{cases} g(u)e^{-\theta(a)(u-x)}, & \text{if } x < u; \\ g(x), & \text{if } x \geq u; \end{cases} \tag{6.1}$$

where $\theta(a) = \sqrt{a^2 + 2\gamma} + a$. (Note that $\theta(a)$ equals $\Phi(\gamma)$ in (3.2) when $\{X_t\}$ is BM(a) and that (6.1) reduces to (5.1) since $\theta(a) = 2a$ for $\gamma = 0$ and $a > 0$.) Clearly with $\gamma > 0$ fixed, $\theta(a)$ is continuous and strictly increasing with $\lim_{a\to-\infty} \theta(a) = 0$ and $\lim_{a\to\infty} \theta(a) = \infty$. The following result can be established along the lines of the proof of Theorem 5.1 with $2a$ replaced by $\theta(a)$ and some minor modifications.

Theorem 6.1. *Fix $\gamma > 0$. Let $g : \mathbb{R} \to [0, \infty)$ be a measurable reward function. Suppose that for each $-\infty < a < \infty$, there is a threshold $u(a) \in \mathbb{R} \cup \{-\infty\}$ such that $\tau_{u(a)}$ is optimal under BM(a). Then g is increasing and logconcave.*

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References

[1] Baurdoux, E.J. (2007). Examples of optimal stopping via measure transformation for processes with one-sided jumps. *Stochastics* **79**, 303–307.

[2] Bertoin, J. (1996). *Lévy Processes.* Cambridge University Press.

[3] Brown, M., Peköz, E.A. and Ross, S.M. (2010). Some results for skip-free random walk. *Probability in the Engineering and Informational Sciences* **24**, 491–507.

[4] Chow, Y.S., Robbins, H. and Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Houghton Mifflin, Boston.

[5] Christensen, S., Irle, A. and Novikov, A. (2011). An elementary approach to optimal stopping problems for AR(1) sequences. *Sequential Analysis* **30**, 79–93.

[6] Christensen, S., Salminen, P. and Ta, B.Q. (2013). Optimal stopping of strong Markov processes. *Stochastic Processes and their Applications* **123**, 1138–1159.

[7] Darling, D.A., Liggett, T. and Taylor, H.M. (1972). Optimal stopping for partial sums. *Ann. Math. Statist.* **43,** 1363–1368.

[8] Deligiannidis, G., Le, H. and Utev, S. (2009). Optimal stopping for processes with independent increments, and applications. *J. Appl. Probab.* **46**, 1130–1145.

[9] Dubins, L.E. and Teicher, H. (1967). Optimal stopping when the future is discounted. *Ann. Math. Statist.* **38,** 601–605.

[10] Kyprianou, A.E. and Palmowski, Z. (2005). A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857,** 16–29. Springer, Berlin.

[11] Kyprianou, A.E. and Surya, B.A. (2005). On the Novikov-Shiryaev optimal stopping problems in continuous time. *Elect. Comm. in Probab.* **10**, 146–154.

[12] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance and Stochastics* **6**, 473–493.

[13] Novikov, A.A. and Shiryaev, A.N. (2005). On an effective solution to the optimal stopping problem for random walks. *Theory. Probab. Appl.* **48,** 288–303.

[14] Novikov, A.A. and Shiryaev, A.N. (2007). On a solution of the optimal stopping problem for processes with independent increments. *Stochastics.* **79,** 393–406.

[15] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press.

[16] Surya, B.A. (2007). An approach for solving perpetual optimal stopping problems driven by Lévy processes. *Stochastics.* **79,** 337–361.